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Digital Signal Processing-18EC52

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Module 1

Discrete Fourier Transform

Signal: a signal is a function that conveys some information
Processing: is changing the signal or extracting information from the signal

All signals are analog in nature in real time. Analog signals are sampled to derive digital signals.
Signals exist in time domain. These have to be transformed into the frequency domain using any of the transformation techniques.
Processing of the signal is done in the frequency domain.
The frequency domain signal is converted back to time domain.

Ex: consider, a sound signal may be a music signal. The signal is analog, which is sampled to get a digital signal.
The next step is to convert the time domain digital signal to frequency domain signal.
Then the signal is subjected to processing may be filtering.
The signal is converted back to the time domain.

Digital signal processing consists of sampling an analog signal, transforming it and then processing it.

Discrete Fourier Transform

Introduction:
A discrete time sysytem may be described by the convolution sum, the fourier representation and the z transform as seen in the previous chapter.
If the signal is periodic in the time domain DTFS representation can be used, in the frequency domain the spectrum is discrete and periodic.If the signal is non-periodic or of finite duration the frequency domain representation is periodic and continuous which is not convenient to implement on the computer. Exploiting the periodicity property of DTFS representation the finite duration sequence can also be represented in the frequency domain, which is referred to as Discrete Fourier Transform DFT.
DFT is an important mathematical tool which can be used for the software implementation of certain digital signal processing algorithms .DFT gives a method to

transform a given sequence to frequency domain and to represent the spectrum of the sequence using only k frequency values, where k is an integer that takes N values,

K=0,1,2,…..N-1.

The advantage of DFT are :

1. It is computationally convenient.
2. The DFT of a finite length sequence makes the frequency domain analysis much simpler than continuous Fourier transform technique.

Discrete Fourier Transform:

The DTFT representation for a finite duration sequence is

$$X(j\omega)=\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$X(n)=1/2\pi \int_{2\pi} X(j\omega)e^{j\omega n}\, d\omega \qquad \omega= 2\pi k/n$$

where x(n) is a finite duration sequence, $X(j\omega)$ is peiodic with period 2π.It is convenient sample $X(j\omega)$ with a sampling frequency equal an integer multiple of its period =m that is taking N uniformly spaced samples between 0 and 2π.

Let $\omega_k= 2\pi k/n$, $0\le k\le N-1$

$$\text{Therefore } X(j\omega)=\sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi kn/N}$$

Since $X(j\omega)$ is sampled for one period and there are N samples $X(j\omega)$ can be expressed as

$$X(k)= X(j\omega)\Big|_{\omega=2\pi kn/N} =\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad 0\le k\le N-1$$

Matrix relation of DFT

The DFT expression can be expressed as

[X] = [x(n)] [WN]

where $[X] = [X(0), X(1),\ldots\ldots]^T$

[x] is the transpose of the input sequence. WN is a N x N matrix

WN = 1 1 1 1 ……1
1 wn1 wn2 wn3….wn n-1
1 wn2 wn4 wn6 …wn2(n-1)
……………………………………

……………………………

…………………………………….

1 …………………………..wN (N-1)(N-1)

ex;
4 pt DFT of the sequence 0,1,2,3

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Solving the matrix X(K) = 6 , -2+2j, -2 , -2-2j

Module-2

Discrete Fourier Transform and its properties

Linearity:

A x1 (n) + b x2 (n) ⟵⟶ a X1(k) + b X2(k)

Circular shift:

In linear shift, when a sequence is shifted the sequence gets extended. In circular shift the number of elements in a sequence remains the same. Given a sequence x (n) the shifted version x(n-m) indicates a shift of m. With dfts the sequences are defined for 0 to N-1.

If x(n) = x(0), x(1), x(2), x(3)

X(n-1) = x(3), x(0),x(1).x(2)

X(n-2) = x(2), x(3), x(0), x(1)

Time shift thm:

If x(n) ⟵⟶ X(k)

Then x(n-m) ⟵⟶ WN^{mk} X(k)

Frequency shift

If x(n)⟵⟶ X(k)

Wn^{+nok} x(n) ⟵⟶X(k+no)

N-1 kn

Consider $x(k) = \sum_{n=0}^{N-1} x(n)\, W^{n}$

$$X(k+no) = \sum_{n=0}^{N-1} x(n)\, WN^{(k+no)n}$$

$$= \sum x(n)\, WN^{kn}\, WN^{non}$$

$\therefore\ X(k+no) \longleftrightarrow x(n)\, WN^{non}$

Modu

Symmetry:

For a real sequence, if $x(n) \longleftrightarrow X(k)$

$$X(N-K) = X^*(k)$$

For a complex sequence
$DFT(x^*(n)) = X^*(N-K)$

If x(n)	then X(k)
Real and even	real and even
Real and odd	imaginary and odd
Odd and imaginary	real odd
Even and imaginary	imaginary and even

convolution theorem;

Circular convolution in time domain corresponds to multiplication of the DFTs

If $y(n) = x(n) \otimes h(n)$ then $Y(k) = X(k)\, H(k)$

Ex let $x(n) = 1,2,2,1$ and $h(n) = 1,2,2,1$
Then $y(n) = x(n) \otimes h(n)$

$Y(n) = 9,10,9,8$

<u>2N pt DFTs of 2 real sequences can be found using a single DF</u>T

If $g(n)$ & $h(n)$ are two sequences then let $x(n) = g(n) + j\, h(n)$

$G(k) = ½\,(X(k) + X^*(k))$

$H(k) = 1/2j\,(X(K) + X^*(k))$

<u>2N pt DFT of a real sequence using a single N pt DFT</u>

let $x(n)$ be a real sequence of length 2N with $y(n)$ and $g(n)$ denoting its N pt dft

let $y(n) = x(2n)$ and $g(2n+1)$

$X(k) = Y(k) + W_N^k\, G(k)$

Using DFT to find IDFT

The DFT expression can be used to find IDFT

$X(n) = 1/N\, [DFT(X^*(k)]^*$

<u>Digital filtering using DFT</u>

In a lti system the system response is got by convoluting the input with the impulse response. In the frequency domain their respective spectra are multiplied. These spectra are continuous and hence cannot be used for computations. The product of 2 DFT s is equivalent to the circular convolution of the corresponding time domain sequences. Circular convolution cannot be used to determine the output of a linear filter to a given input sequence. In this case a frequency domain methodology equivalent to linear convolution is required. Linear convolution can be implemented using circular convolution by taking the length of the convolution as N >= n1+n2-1 where n1 and n2 are the lengths of the 2 sequences.

Overlap and add

In order to convolve a short duration sequence with a long duration sequence x(n) ,x(n) is split into blocks of length N x(n) and h(n) are zero padded to length N+K-1 . circular convolution is performed to each block then the results are added.

Overlap and save method

In this method x(n) is divided into blocks of length N with an overlap of k-1 samples. the first block is zero padded with k-1 zeros at the beginning. H(n) is also zero padded to length N . circular convolution of each block is performed using the N length DFT .The output signal is obtained after discarding the first k-1 samples The final result is obtained by adding the intermediate results.

Digital Signal Processing-18EC52

Module-3

Fast Fourier Transform (FFT)

In this section we present several methods for computing the DFT efficiently. In view of the importance of the DFT in various digital signal processing applications, such as linear filtering, correlation analysis, and spectrum analysis, its efficient computation is a topic that has received considerable attention by many mathematicians, engineers, and applied scientists.

From this point, we change the notation that $X(k)$, instead of $y(k)$ in previous sections, represents the Fourier coefficients of $x(n)$.

Basically, the computational problem for the DFT is to compute the sequence $\{X(k)\}$ of N complex-valued numbers given another sequence of data $\{x(n)\}$ of length N, according to the formula

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad 0 \le k \le N-1$$

$$W_N = e^{-j2\pi/N}$$

In general, the data sequence $x(n)$ is also assumed to be complex valued. Similarly, The IDFT becomes

$$x(n) = \frac{1}{N}\sum_{n=0}^{N-1} X(k) W_N^{-nk}, \qquad 0 \le n \le N-1$$

Since DFT and IDFT involve basically the same type of computations, our discussion of efficient computational algorithms for the DFT applies as well to the efficient computation of the IDFT.

We observe that for each value of k, direct computation of $X(k)$ involves N complex multiplications ($4N$ real multiplications) and N-1 complex additions ($4N$-2 real additions). Consequently, to compute all N values of the DFT requires N^2 complex multiplications and N^2-N complex additions.

Direct computation of the DFT is basically inefficient primarily because it does not exploit the symmetry and periodicity properties of the phase factor W_N. In particular, these two properties are :

$$\text{Symmetry property} : W_N^{k+N/2} = -W_N^k$$
$$\text{Periodicity property} : W_N^{k+N} = W_N^k$$

The computationally efficient algorithms described in this section, known collectively as fast Fourier transform (FFT) algorithms, exploit these two basic properties of the phase factor.

Radix-2 FFT Algorithms

Let us consider the computation of the $N = 2^v$ point DFT by the divide-and conquer approach. We split the N-point data sequence into two $N/2$-point data sequences $f_1(n)$ and $f_2(n)$, corresponding to the even-numbered and odd-numbered samples of $x(n)$, respectively, that is,

$$f_1(n) = x(2n)$$
$$f_2(n) = x(2n+1), \qquad n = 0,1,\ldots,\frac{N}{2}-1$$

Thus $f_1(n)$ and $f_2(n)$ are obtained by decimating $x(n)$ by a factor of 2, and hence the resulting FFT algorithm is called a *decimation-in-time algorithm.*

Now the N-point DFT can be expressed in terms of the DFT's of the decimated sequences as follows:

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0,1,\ldots,N-1 \\ &= \sum_{n \text{ even}} x(n) W_N^{kn} + \sum_{n \text{ odd}} x(n) W_N^{kn} \\ &= \sum_{m=0}^{(N/2)-1} x(2m) W_N^{2mk} + \sum_{m=0}^{(N/2)-1} x(2m+1) W_N^{k(2m+1)} \end{aligned}$$

But $W_N^2 = W_{N/2}$. With this substitution, the equation can be expressed as

$$X(k) = \sum_{m=0}^{(N/2)-1} f_1(m) W_{N/2}^{km} + W_N^k \sum_{m=0}^{(N/2)-1} f_2(m) W_{N/2}^{km}$$
$$= F_1(k) + W_N^k F_2(k), \qquad k = 0,1,\ldots,N-1$$

where $F_1(k)$ and $F_2(k)$ are the $N/2$-point DFTs of the sequences $f_1(m)$ and $f_2(m)$, respectively.

Since $F_1(k)$ and $F_2(k)$ are periodic, with period $N/2$, we have $F_1(k+N/2) = F_1(k)$ and $F_2(k+N/2) = F_2(k)$. In addition, the factor $W_N^{k+N/2} = -W_N^k$. Hence the equation may be expressed as

$$X(k) = F_1(k) + W_N^k F_2(k), \qquad k = 0,1,\ldots,\frac{N}{2}-1$$
$$X(k+\frac{N}{2}) = F_1(k) - W_N^k F_2(k), \qquad k = 0,1,\ldots,\frac{N}{2}-1$$

We observe that the direct computation of $F_1(k)$ requires $(N/2)^2$ complex multiplications. The same applies to the computation of $F_2(k)$. Furthermore, there are $N/2$ additional complex multiplications required to compute $W_N^k F_2(k)$. Hence the computation of $X(k)$ requires $2(N/2)^2 + N/2 = N^2/2 + N/2$ complex multiplications. This first step results in a reduction of the number of multiplications from N^2 to $N^2/2 + N/2$, which is about a factor of 2 for N large.

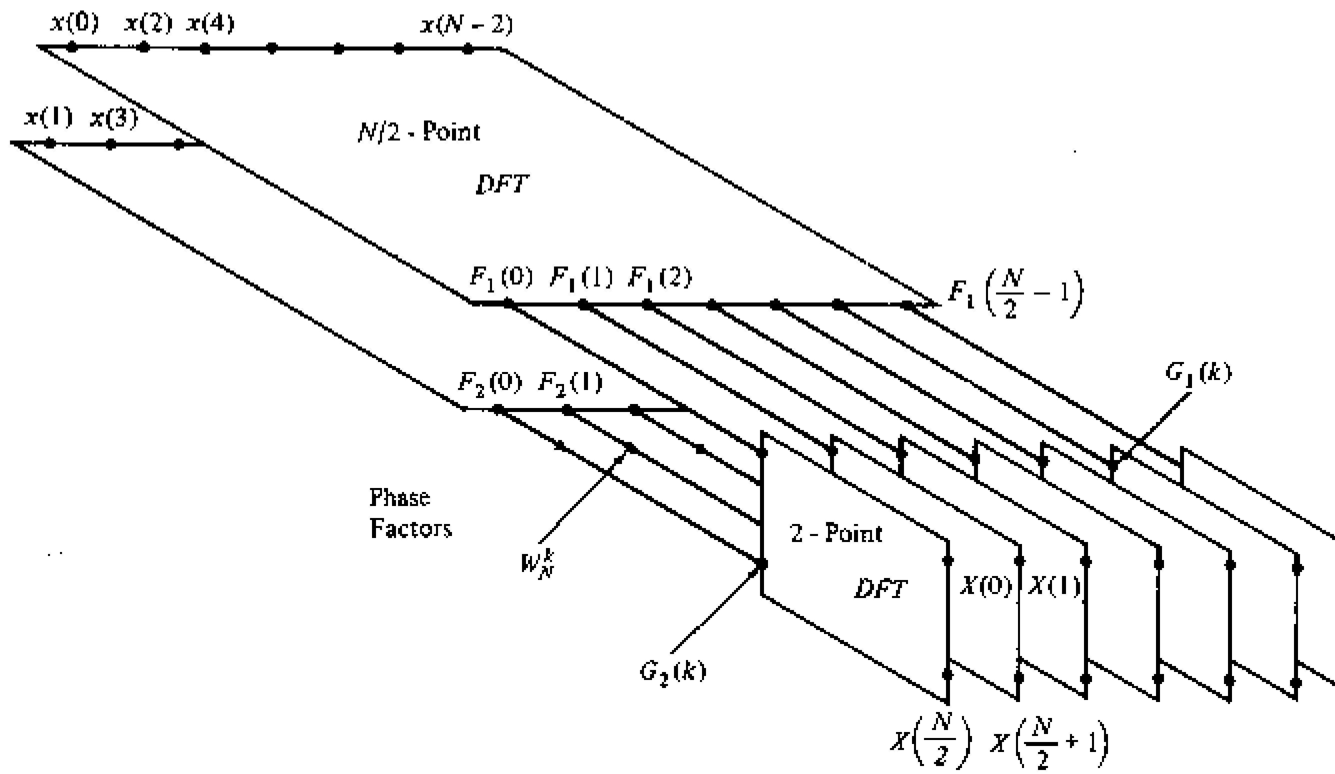


Figure TC.3.1 First step in the decimation-in-time algorithm.

By computing $N/4$-point DFTs, we would obtain the $N/2$-point DFTs $F_1(k)$ and $F_2(k)$ from the relations

$$F_1(k) = \mathrm{F}\{f_1(2n)\} + W_{N/2}^k\,\mathrm{F}\{f_1(2n+1)\}, \qquad k = 0,1,\ldots,\frac{N}{4}-1; \qquad n = 0,1,\ldots,\frac{N}{4}-1$$

$$F_1\left(k+\frac{N}{4}\right) = \mathrm{F}\{f_1(2n)\} - W_{N/2}^k\,\mathrm{F}\{f_1(2n+1)\}, \qquad k = 0,1,\ldots,\frac{N}{4}-1; \qquad n = 0,1,\ldots,\frac{N}{4}-1$$

$$F_2(k) = \mathrm{F}\{f_2(2n)\} + W_{N/2}^k\,\mathrm{F}\{f_2(2n+1)\}, \qquad k = 0,1,\ldots,\frac{N}{4}-1; \qquad n = 0,1,\ldots,\frac{N}{4}-1$$

$$F_2\left(k+\frac{N}{4}\right) = \mathrm{F}\{f_2(2n)\} - W_{N/2}^k\,\mathrm{F}\{f_2(2n+1)\}, \qquad k = 0,1,\ldots,\frac{N}{4}-1; \qquad n = 0,1,\ldots,\frac{N}{4}-1$$

$\mathrm{F}\{*\}$ represents Fourier transform

The decimation of the data sequence can be repeated again and again until the resulting sequences are reduced to one-point sequences. For $N = 2^v$, this decimation can be performed $v = \log_2 N$ times. Thus the total number of complex multiplications is reduced to $(N/2)\log_2 N$. The number of complex additions is $N\log_2 N$.

For illustrative purposes, Figure TC.3.2 depicts the computation of $N = 8$ point DFT. We observe that the computation is performed in tree stages, beginning with the computations of four two-point DFTs, then two four-point DFTs, and finally, one eight-point DFT. The combination for the smaller DFTs to form the larger DFT is illustrated in Figure TC.3.3 for $N = 8$.

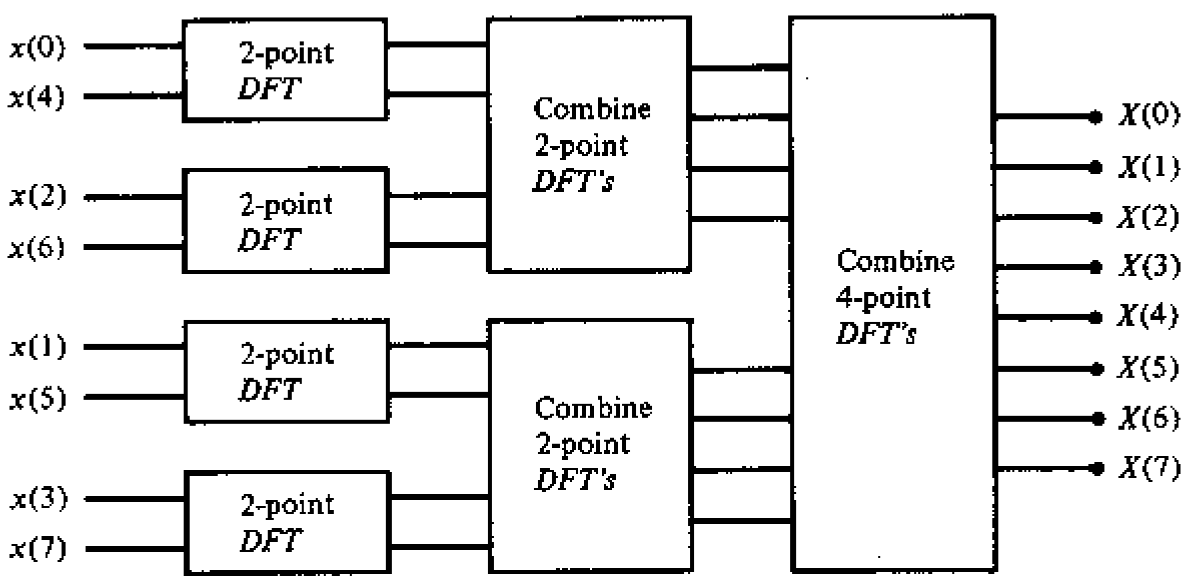


Figure TC.3.2 Three stages in the computation of an N = 8-point DFT.

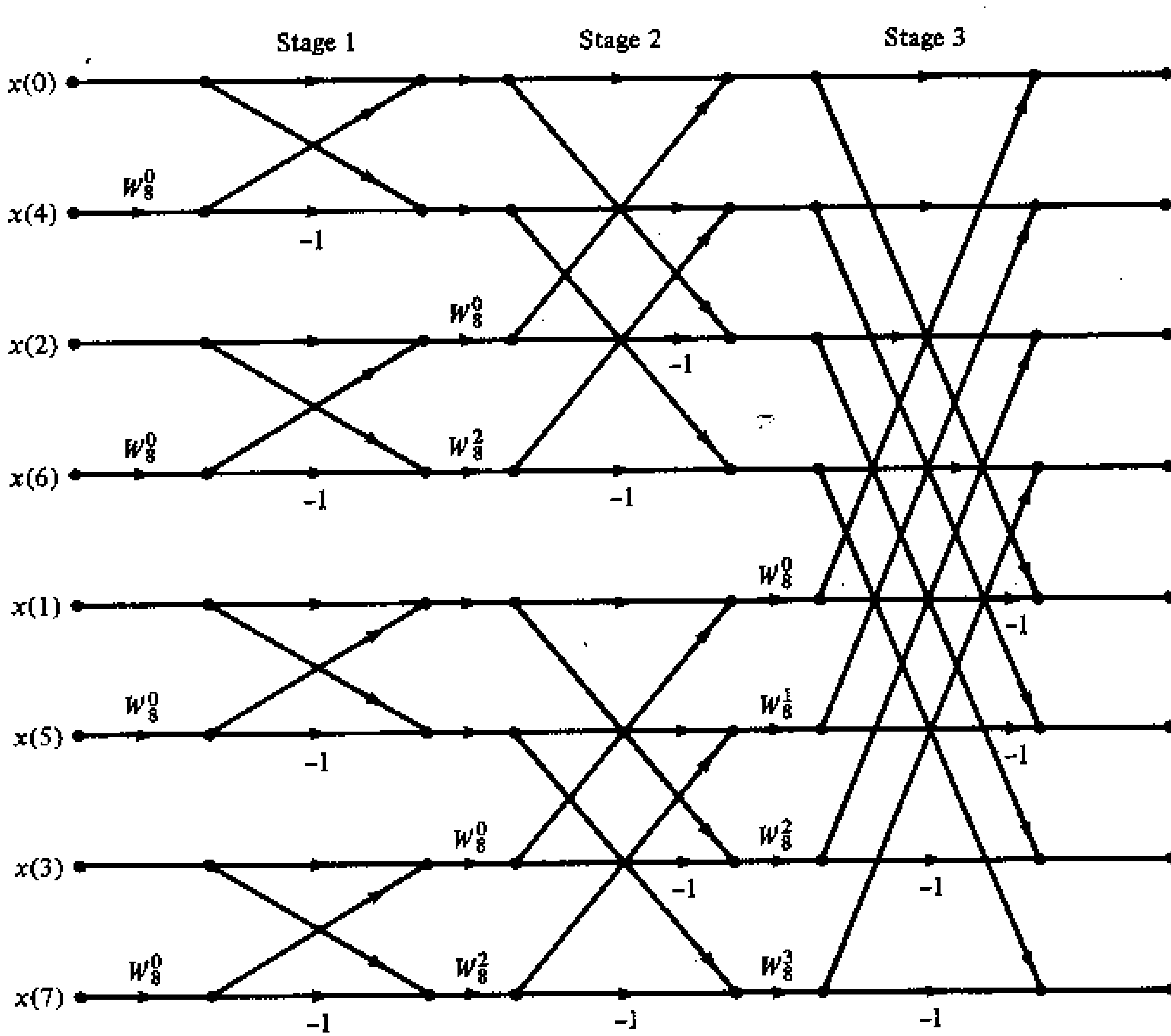


Figure TC.3.3 Eight-point decimation-in-time FFT algorithm.

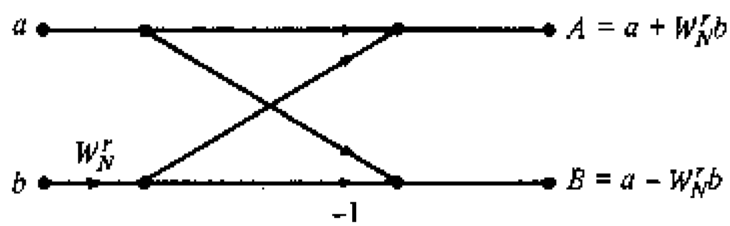
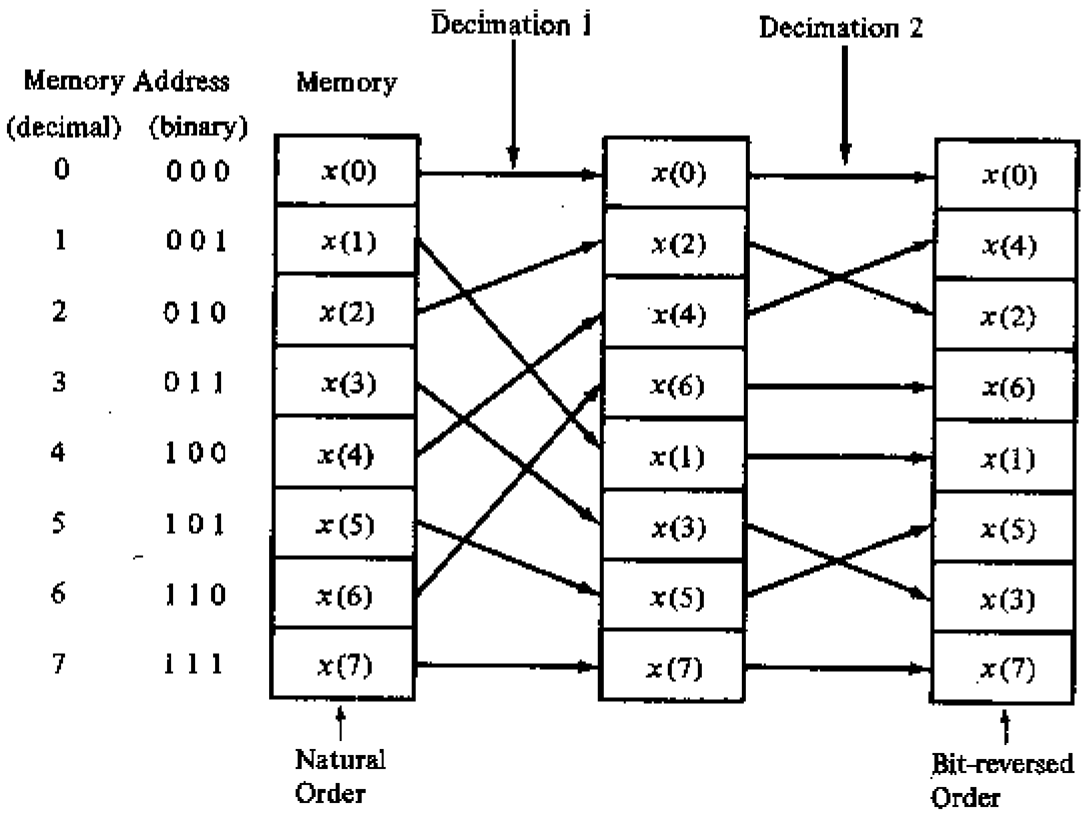


Figure TC.3.4 Basic butterfly computation in the decimation-in-time FFT algorithm.

An important observation is concerned with the order of the input data sequence after it is decimated (v-1) times. For example, if we consider the case where N = 8, we know that the first decimation yeilds the sequence *x(0), x(2), x(4), x(6), x(1), x(3), x(5), x(7),* and the second decimation results in the sequence *x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7).* This *shuffling* of the input data sequence has a well-defined order as can be ascertained from observing Figure TC.3.5, which illustrates the decimation of the eight-point sequence.



(a)

$(n_2 n_1 n_0)$	→	$(n_0 n_2 n_1)$	→	$(n_0 n_1 n_2)$
(0 0 0)	→	(0 0 0)	→	(0 0 0)
(0 0 1)	→	(1 0 0)	→	(1 0 0)
(0 1 0)	→	(0 0 1)	→	(0 1 0)
(0 1 1)	→	(1 0 1)	→	(1 1 0)
(1 0 0)	→	(0 1 0)	→	(0 0 1)
(1 0 1)	→	(1 1 0)	→	(1 0 1)
(1 1 0)	→	(0 1 1)	→	(0 1 1)
(1 1 1)	→	(1 1 1)	→	(1 1 1)

(b)

Figure TC.3.5 Shuffling of the data and bit reversal.

Another important radix-2 FFT algorithm, called the decimation-in-frequency algorithm, is obtained by using the divide-and-conquer approach. To derive the algorithm, we begin by splitting the DFT formula into two summations, one of which involves the sum over the first $N/2$ data points and the second sum involves the last N/2 data points. Thus we obtain

$$\begin{aligned} X(k) &= \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} \\ &= \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + W_N^{Nk/2} \sum_{n=0}^{(N/2)-1} x\left(n+\frac{N}{2}\right) W_N^{kn} \end{aligned}$$

Since $W_N^{kN/2} = (-1)^k$

$$X(k) = \sum_{n=0}^{(N/2)-1} \left[x(n) + (-1)^k x\left(n+\frac{N}{2}\right) \right] W_N^{kn}$$

Now, let us split (decimate) X(k) into the even- and odd-numbered samples. Thus we obtain

$$X(2k) = \sum_{n=0}^{(N/2)-1} \left[x(n) + x\left(n+\frac{N}{2}\right) \right], \qquad k = 0,1,\ldots,\frac{N}{2}-1$$

$$X(2k+1) = \sum_{n=0}^{(N/2)-1} \left\{ \left[x(n) - x\left(n+\frac{N}{2}\right) \right] \right\}, \qquad k = 0,1,\ldots,\frac{N}{2}-1$$

where we have used the fact that $W_N^{\,2} = W_{N/2}$

The computational procedure above can be repeated through decimation of the $N/2$-point DFTs $X(2k)$ and $X(2k+1)$. The entire process involves v = $\log_2 N$ stages of decimation, where each stage involves $N/2$ butterflies of the type shown in Figure TC.3.7. Consequently, the computation of the N-point DFT via the decimation-in-frequency FFT requires $(N/2)\log_2 N$ complex multiplications and $N\log_2 N$ complex additions, just as in the decimation-in-time algorithm. For illustrative purposes, the eight-point decimation-in-frequency algorithm is given in Figure TC.3.8.

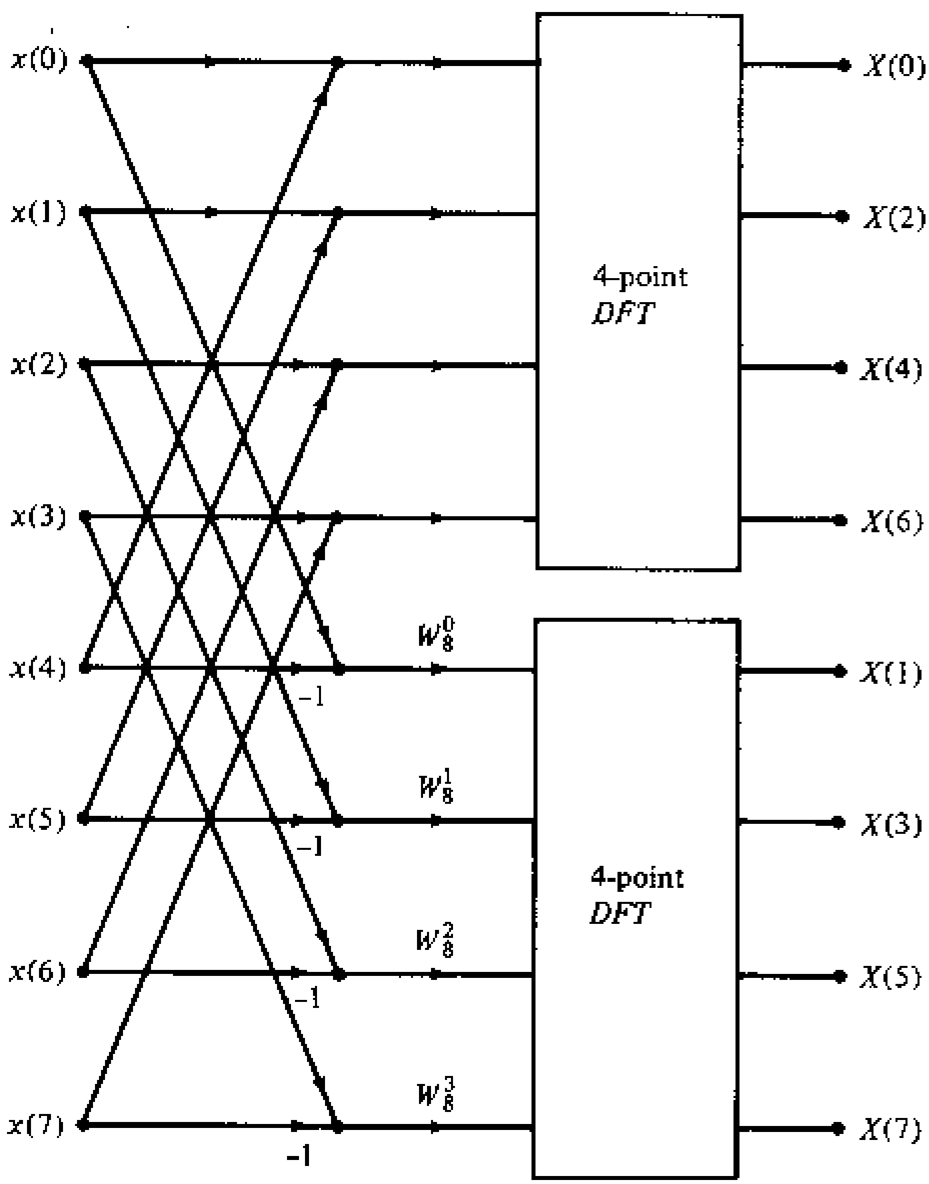


Figure TC.3.6 First stage of the decimation-in-frequency FFT algorithm.

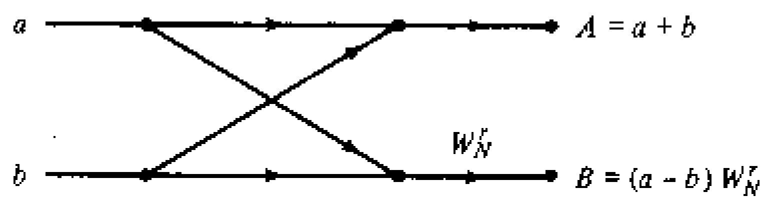


Figure TC.3.7 Basic butterfly computation in the decimation-in-frequency.

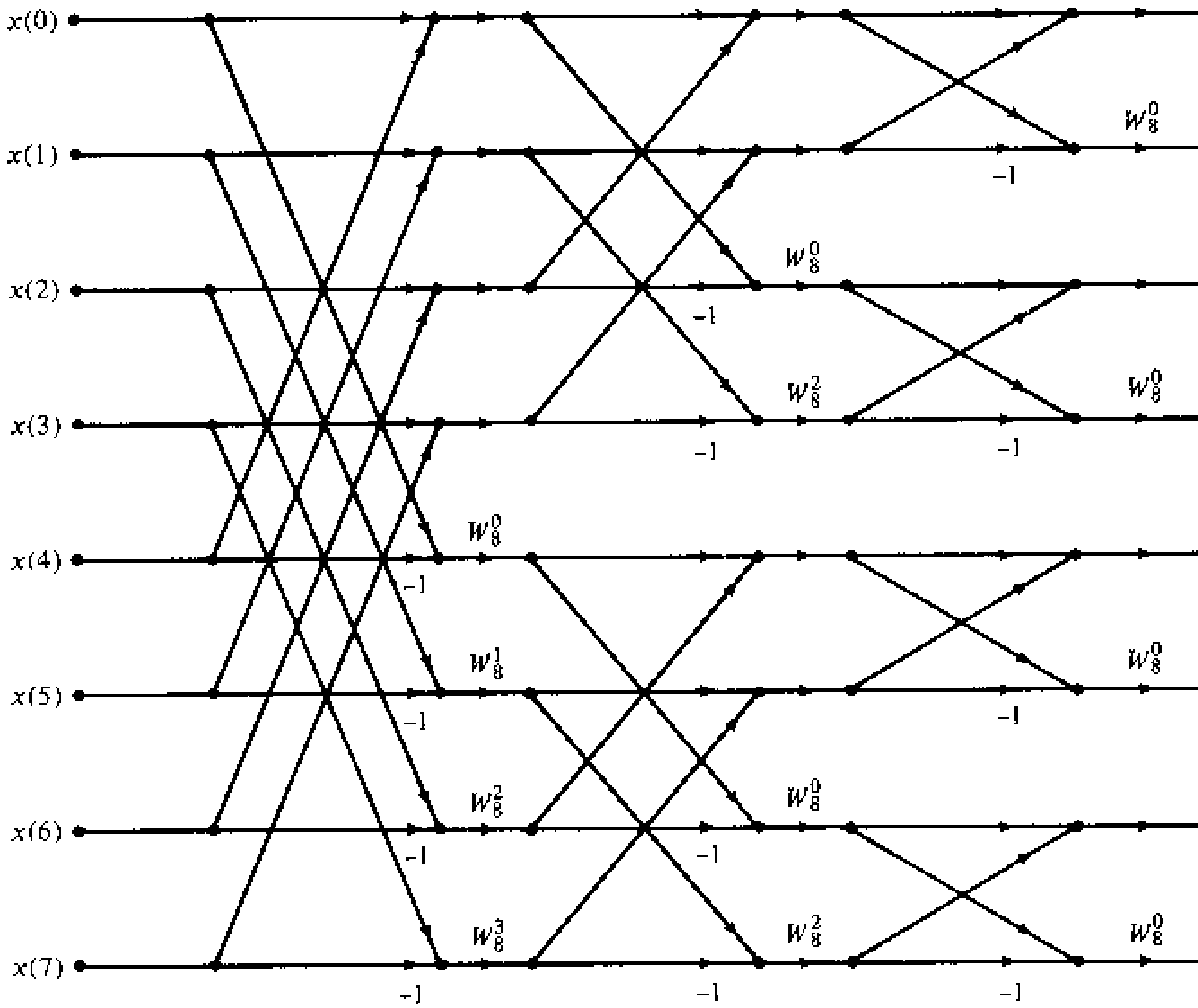


Figure TC.3.8 N = 8-piont decimation-in-frequency FFT algorithm.

We observe from Figure TC.3.8 that the input data $x(n)$ occurs in natural order, but the output DFT occurs in bit-reversed order. We also note that the computations are performed in place. However, it is possible to reconfigure the decimation-in-frequency algorithm so that the input sequence occurs in bit-reversed order while the output DFT occurs in normal order. Furthermore, if we abandon the requirement that the computations be done in place, it is also possible to have both the input data and the output DFT in normal order.

	Real Multiplications				**Real Additions**			
N	**Radix-2**	**Radix-4**	**Radix-8**	**Split Radix**	**Radix-2**	**Radix-4**	**Radix-8**	**Split Radix**
16	24	20		20	152	148		148
32	88			68	408			388
64	264	208	204	196	1032	976	972	964
128	72			516	2054			2308
256	1800	1392		1284	5896	5488		5380
512	4360		3204	3076	13566		12420	12292
1024	10248	7856		7172	30728	28336		27652

Table TC.3.1 Number of Nontrivial Real Multiplcations and Additions to Compute an *N*-point Complex DFT

UNIT_4

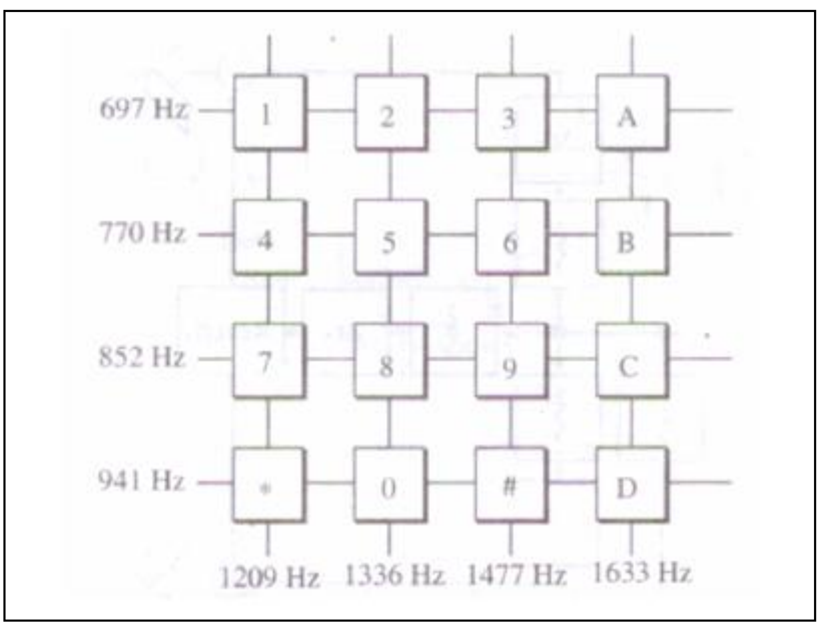
Goertzel's Algorithm

1.1 Introduction:

Standard frequency analysis requires transforming time-domain signal to frequency domain and studying Spectrum of the signal. This is done through DFT computation. N-point DFT computation results in N frequency components. We know that DFT computation through FFT requires N/2 $\log_2 N$ complex multiplications and N $\log_2 N$ additions. In certain applications not all N frequency components need to be computed (an application will be discussed). If the desired number of values of the DFT is less than 2 $\log_2 N$ than direct computation of the desired values is more efficient that FFT based computation.

1.2 Example: DTMF – Dual Tone Multifrequency

This is known as touch-tone/speed/electronic dialing, pressing of each button generates a unique set of two-tone signals, called DTMF signals. These signals are processed at exchange to identify the number pressed by determining the two associated tone frequencies. Seven frequencies are used to code the 10 decimal digits and two special characters (4x3 array)



In this application frequency analysis requires determination of possible seven (eight) DTMF fundamental tones and their respective second harmonics .For an 8 kHz sampling freq, the best value of the DFT length N to detect the eight fundamental DTMF tones has been found to be 205 .Not all 205 freq components are needed here, instead only those corresponding to key frequencies are required.FFT algorithm is not effective and efficient in this application. The direct computation of the DFT which is more effective in this application is formulated as a linear filtering operation on the input data sequence.

This algorithm is known as Goertzel Algorithm

This algorithm exploits periodicity property of the phase factor. Consider the DFT definition

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \qquad (1)$$

Since W_N^{-kN} ***is equal to 1, multiplying both sides of the equation by this results in;***

$$X(k) = W_N^{-kN} \sum_{m=0}^{N-1} x(m) W_N^{mk} = \sum_{m=0}^{N-1} x(m) W_N^{-k(N-m)} \qquad (2)$$

This is in the form of a convolution $y_k(n) = x(n) * h_k(n)$

$$y_k(n) = \sum_{m=0}^{N-1} x(m) W_N^{-k(n-m)} \qquad (3)$$

$$h_k(n) = W_N^{-kn} u(n) \qquad (4)$$

Where $y_k(n)$ is the out put of a filter which has impulse response of $h_k(n)$ and input $x(n)$.

The output of the filter at n = N yields the value of the DFT at the freq $\omega_k = 2\pi k/N$

The filter has frequency response given by

$$H_k(z) = \frac{1}{1 - W_N^{-k} z^{-1}} \qquad (6)$$

The above form of filter response shows it has a pole on the unit circle at the frequency $\omega_k = 2\pi k/N$.

Entire DFT can be computed by passing the block of input data into a parallel bank of N single-pole filters (resonators)

1.3 Difference Equation implementation of filter:

From the frequency response of the filter (eq 6) we can write the following difference equation relating input and output;

$$H_k(z) = \frac{Y_k(z)}{X(z)} = \frac{1}{1 - W_N^{-k} z^{-1}}$$

$$y_k(n) = W_N^{-k} y_k(n-1) + x(n) \qquad y_k(-1) = 0 \qquad (7)$$

The desired output is $X(k) = y_k(n)$ for $k = 0,1,\ldots N-1$. The phase factor appearing in the difference equation can be computed once and stored.

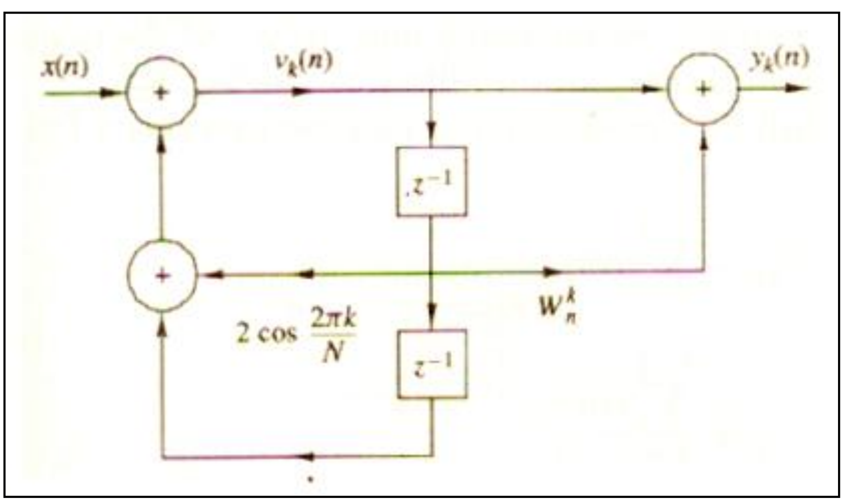
The form shown in eq (7) requires complex multiplications which can be avoided doing suitable modifications(divide and multiply by $1-W_N^k z^{-1}$). Then frequency response of the filter can be alternatively expressed as

$$H_k(z) = \frac{1-W_N^k z^{-1}}{1-2\cos(2\pi k/N)z^{-1}+z^{-2}} \qquad (8)$$

This is second –order realization of the filter (observe the denominator now is a second-order expression). The direct form realization of the above is given by

$$v_k(n) = 2\cos(2\pi k/N)v_k(n-1) - v_k(n-2) + x(n) \qquad (9)$$

$$y_k(n) = v_k(n) - W_N^k v_k(n-1) \qquad v_k(-1) = v_k(-2) = 0 \qquad (10)$$



The recursive relation in (9) is iterated for $n = 0,1,\ldots\ldots N$, but the equation in (10) is computed only once at time $n = N$. Each iteration requires one real multiplication and two additions. Thus, for a real input sequence $x(n)$ this algorithm requires $(N+1)$ real multiplications to yield $X(k)$ and $X(N-k)$ (this is due to symmetry). Going through the Goertzel algorithm it is clear that this algorithm is useful only when M out of N DFT values need to be computed where $M \leq 2\log_2 N$, Otherwise, the FFT algorithm is more efficient method. The utility of the algorithm completely depends on the application and number of frequency components we are looking for.

2. Chirp z- Transform

2.1 Introduction:

Computation of DFT is equivalent to samples of the z-transform of a finite-length sequence at equally spaced points around the unit circle. The spacing between the samples is given by 2π/N. The efficient computation of DFT through FFT requires N to be a highly composite number which is a constraint. Many a times we may need samples of z-transform on contours other than unit circle or we my require dense set of frequency samples over a small region of unit circle. To understand these let us look in to the following situations:

1. ***Obtain samples of z-transform on a circle of radius 'a' which is concentric to unit circle***
   The possible solution is to multiply the input sequence by a^{-n}
2. ***128 samples needed between frequencies***

   $\omega = -\pi/8$ to $+\pi/8$ from a 128 point sequence

   From the given specifications we see that the spacing between the frequency samples is $\pi/512$ or $2\pi/1024$. In order to achieve this freq resolution we take 1024- point FFT of the given 128-point seq by appending the sequence with 896 zeros. Since we need only 128 frequencies out of 1024 there will be big wastage of computations in this scheme.

For the above two problems Chirp z-transform is the alternative.

Chirp z- transform is defined as:

$$X(z_k) = \sum_{n=0}^{N-1} x(n) z_k^{-n} \qquad k = 0,1,......L-1 \qquad (11)$$

Where z_k is a generalized contour.

Z_k is the set of points in the z-plane falling on an arc which begins at some point z_0 and spirals either in toward the origin or out away from the origin such that the points $\{z_k\}$are defined as,

$$z_k = r_0 e^{j\theta_0} (R_0 e^{j\phi_0})^k \qquad k = 0,1,....L-1 \qquad (12)$$

Note that,

a. if $R_0 < 1$ the points fall on a contour that spirals toward the origin

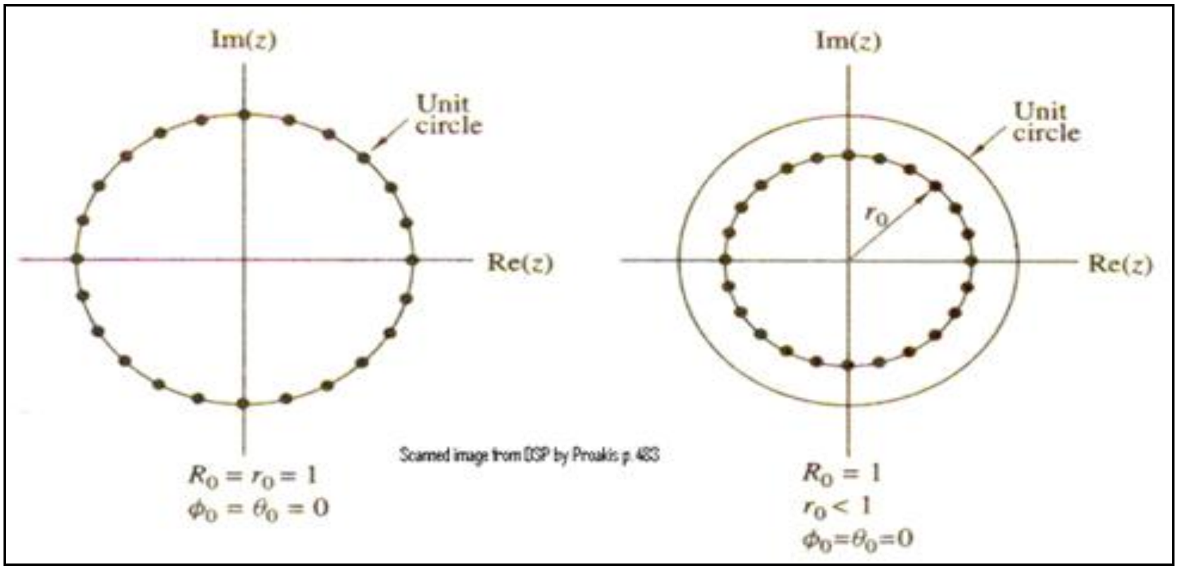
b. If $R_0 > 1$ the contour spirals away from the origin

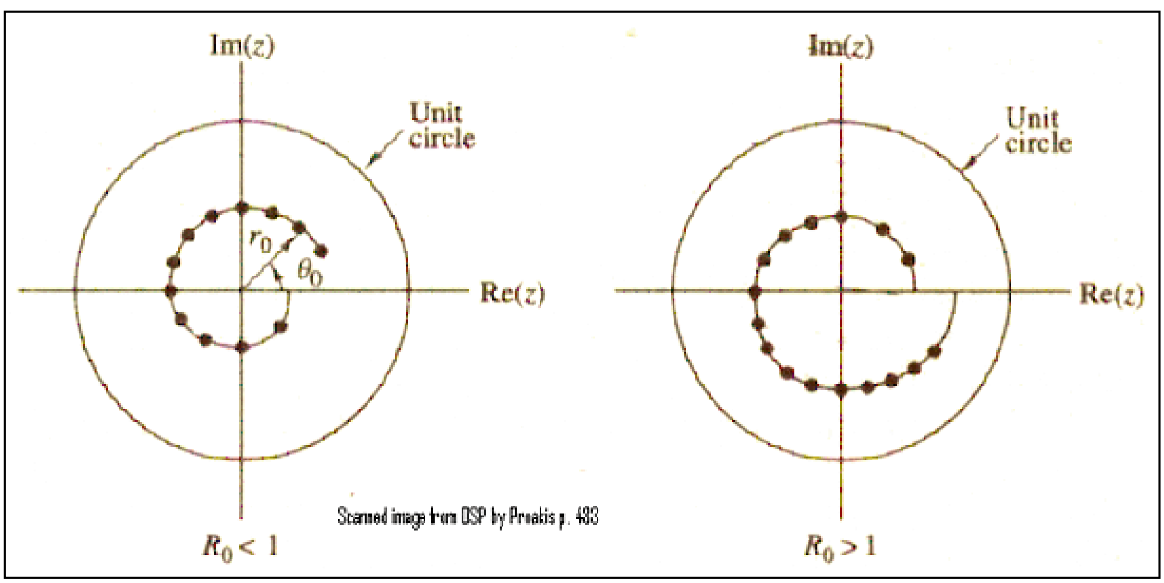
c. If $R_0 = 1$ the contour is a circular arc of radius

d.If $r_0=1$ and $R_0=1$ the contour is an arc of the unit circle.

(Additionally this contour allows one to compute the freq content of the sequence x(n) at dense set of L frequencies in the range covered by the arc without having to compute a large DFT (i.e., a DFT of the sequence x(n) padded with many zeros to obtain the desired resolution in freq.))

e. If $r_0 = R_0=1$ and $\theta_0=0$ $\Phi_0=2\pi/N$ and L = N the contour is the entire unit circle similar to the standard DFT. These conditions are shown in the following diagram.


Im(z)
Unit circle
Re(z)
$R_0 = r_0 = 1$
$\phi_0 = \theta_0 = 0$
Scanned image from DSP by Proakis p. 483
Im(z)
Unit circle
r_0
Re(z)
$R_0 = 1$
$r_0 < 1$
$\phi_0 = \theta_0 = 0$


Im(z)
Unit circle
r_0
θ_0
Re(z)
$R_0 < 1$
Scanned image from DSP by Proakis p. 483
Im(z)
Unit circle
Re(z)
$R_0 > 1$

Substituting the value of z_k in the expression of $X(z_k)$

$$X(z_k) = \sum_{n=0}^{N-1} x(n) z_k^{-n} = \sum_{n=0}^{N-1} x(n)(r_0 e^{j\theta_0})^{-n} W^{-nk} \qquad (13)$$

where

$$W = R_0 e^{j\phi_0} \qquad (14)$$

2.2 Expressing computation of $X(z_k)$ as linear filtering operation:

By substitution of

$$nk = \frac{1}{2}(n^2 + k^2 - (k-n)^2) \qquad (15)$$

we can express $X(z_k)$ as

$$X(z_k) = W^{-k^2/2} y(k) = y(k)/h(k) \qquad k = 0,1,\ldots\ldots\ldots L-1 \qquad (16)$$

Where

$$g(n) = x(n)(r_0 e^{j\theta_0})^{-n} W^{-n^2/2} \qquad h(n) = W^{n^2/2}$$

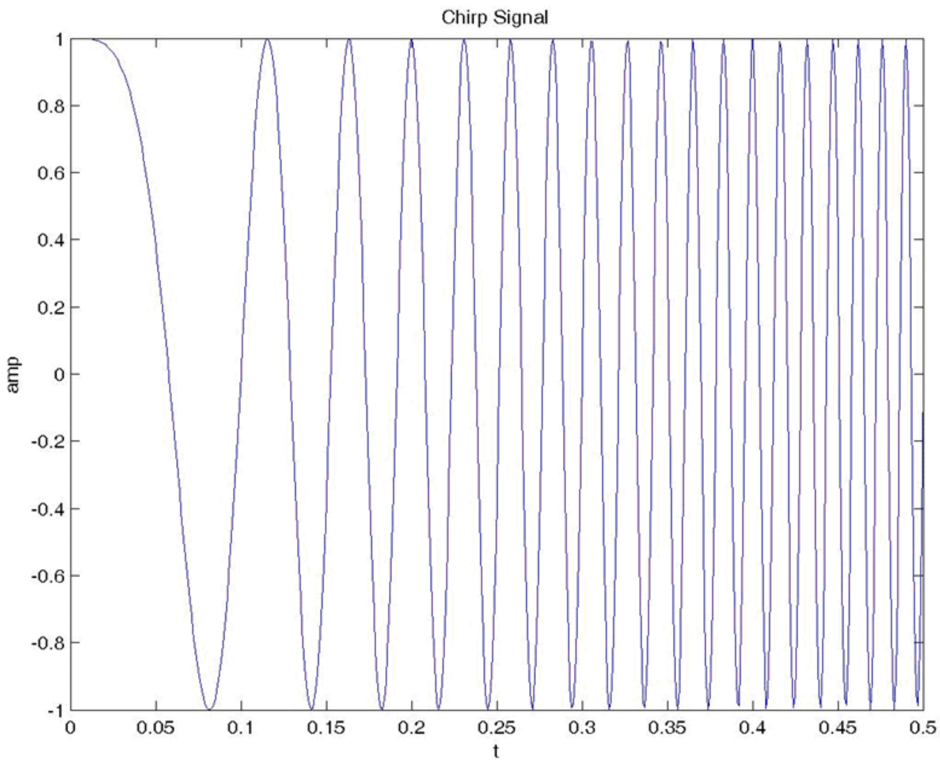
$$y(k) = \sum_{n=0}^{N-1} g(n)h(k-n) \qquad (17)$$

both g(n) and h(n) are complex valued sequences

2.3 Why it is called Chirp z-transform?

if R_0 =1, then seq h(n) has the form of complex exponential with argument

$\omega n = n^2\Phi_0/2 = (n\ \Phi_0/2)\ n$. The quantity $(n\ \Phi_0/2)$ represents the freq of the complez exponential signal, which increases linearly with time. Such signals are used in radar systems are called chirp signals. Hence the name chirp z-transform.



2.4 How to Evaluate linear convolution of eq (17)

1. ***Can be done efficiently with FFT***
2. ***The two sequences involved are g(n) and h(n). g(n) is finite length seq of length N and h(n) is of infinite duration, but fortunately only a portion of h(n) is required to compute L values of X(z), hence FFT could be still be used.***
3. ***Since convolution is via FFT, it is circular convolution of the N-point seq g(n) with an M- point section of h(n) where M > N***
4. ***The concepts used in over lap –save method can be used***
5. ***While circular convolution is used to compute linear convolution of two sequences we know the initial N-1 points contain aliasing and the remaining points are identical to the result that would be obtained from a linear convolution of h(n) and g(n), In view of this the DFT size selected is M = L+N-1 which would yield L valid points and N-1 points corrupted by aliasing. The section of h(n) considered is for –(N-1) ≤ n≤ (L-1) yielding total length M as defined***
6. ***The portion of h(n) can be defined in many ways, one such way is,***

   $h_1(n) = h(n-N+1) \quad n = 0,1,\ldots M-1$
7. ***Compute*** $H_1(k)$ ***and G(k) to obtain***

   $Y_1(k) = G(K)H_1(k)$
8. ***Application of IDFT will give*** $y_1(n)$***, for***

   $n = 0,1,\ldots M-1$***. The starting N-1 are discarded and desired values are*** $y_1(n)$ ***for***

   $N-1 \le n \le M-1$ ***which corresponds to the range*** $0 \le n \le L-1$ ***i.e.,***

   $y(n) = y_1(n+N-1) \quad n=0,1,2,\ldots L-1$
9. ***Alternatively*** $h_2(n)$ ***can be defined as***

$$
\begin{aligned}
h_2(n) &= h(n) & 0 \le n \le L-1 \\
&= h(n-(N+L-1)) & L \le n \le M-1
\end{aligned}
$$

10. ***Compute*** $Y_2(k) = G(K)H_2(k)$***, The desired values of*** $y_2(n)$ ***are in the range***

$0 \leq n \leq L-1$ i.e.,

$y(n) = y_2(n) \quad n=0,1,\ldots.L-1$

11. Finally, the complex values $X(z_k)$ are computed by dividing $y(k)$ by $h(k)$

For $k = 0,1,\ldots\ldots L-1$

2.5 Computational complexity

In general the computational complexity of CZT is of the order of $M \log_2 M$ complex multiplications. This should be compared with N.L which is required for direct evaluation. If L is small direct evaluation is more efficient otherwise if L is large then CZT is more efficient.

2.5.1 Advantages of CZT

a. Not necessary to have N =L

b.Neither N or L need to be highly composite

c.The samples of Z transform are taken on a more general contour that includes the unit circle as a special case.

2.6 Example to understand utility of CZT algortithm in freq analysis

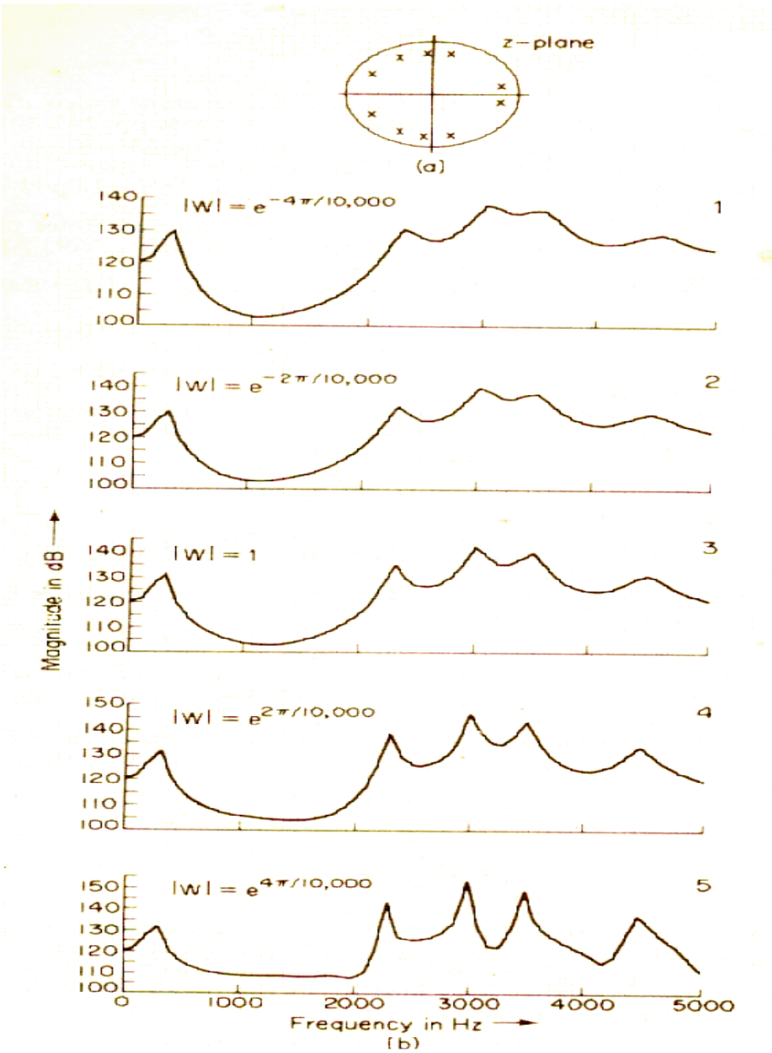
(ref: DSP by Oppenheim Schaffer)

CZT is used in this application to sharpen the resonances by evaluating the z-transform off the unit circle. Signal to be analyzed is a synthetic speech signal generated by exciting a five-pole system with a periodic impulse train. The system was simulated to correspond to a sampling frq of 10khz. The poles are located at center

freqs of 270,2290,3010,3500 & 4500 Hz with bandwidth of 30, 50,60,87 & 140 Hz respectively.

Solution: Observe the pole-zero plots and corresponding magnitude frequency response for different choices of |w|. The following observations are in order:

- ***The first two spectra correspond to spiral contours outside the unit circle with a resulting broadening of the resonance peaks***
- ***|w| = 1 corresponds to evaluating z-transform on the unit circle***
- ***The last two choices corresponds to spiral contours which spirals inside the unit circle and close to the pole locations resulting in a sharpening of resonance peaks.***

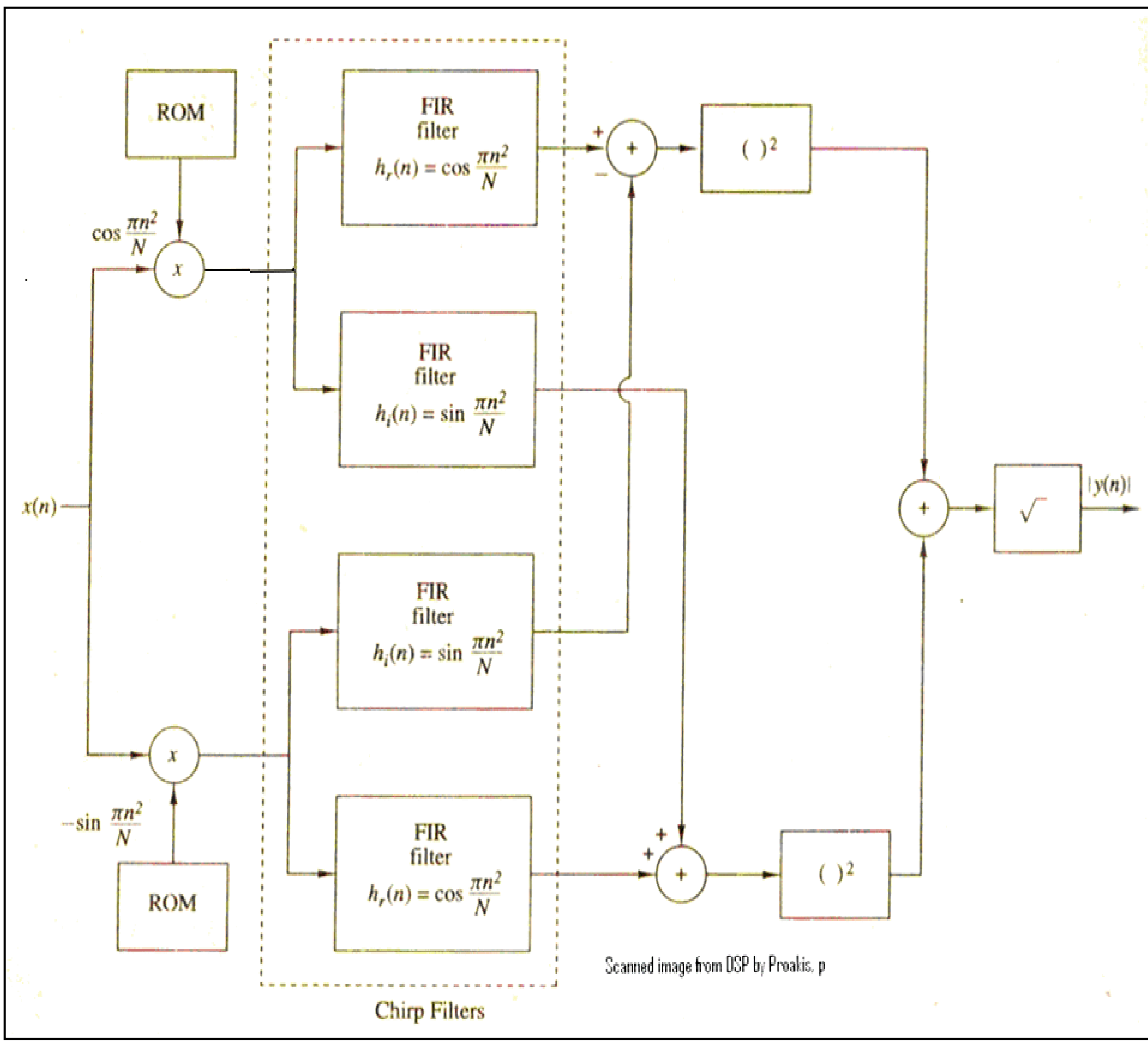

z-plane
(a)
|W| = e−4π/10,000
1
|W| = e−2π/10,000
2
|W| = 1
3
|W| = e2π/10,000
4
|W| = e4π/10,000
5
Magnitude in dB
Frequency in Hz
(b)

2.7 Implementation of CZT in hardware to compute the DFT signals

The block schematic of the CZT hardware is shown in down figure. DFT computation requires $r_0 = R_0 = 1$, $\theta_0 = 0$ $\Phi_0 = 2\pi/N$ and $L = N$.

The cosine and sine sequences in h(n) needed for pre multiplication and post multiplication are usually stored in a ROM. If only magnitude of DFT is desired, the post multiplications are unnecessary,

In this case $|X(z_k)| = |y(k)|$ $k = 0,1,\ldots.N-1$



UNIT5

Frequency Transformation

4.1 Introduction:

Frequency Transformation allows one to design a prototype filter and transform to any specific frequency selective type instead of designing each of the type separately. With frequency transformations designers can concentrate on improved methods of designing prototype rather than wasting time on devising design methodologies for different types of filters. One top of all these one design and all types of frequency selective filters is always an advantage. The techniques of Frequency transformation could be applied in both Analog and Digital domain.

4.2 Frequency Transformation in Analog domain:

In this transformation technique normalized Low Pass filter with cutoff freq of $\Omega_p = 1$ rad/sec is designed and all other types of filters are obtained from this prototype. For example, normalized LPF is transformed to LPF of specific cutoff freq by the following transformation formula,

Normalized LPF to LPF of specific cutoff:

$$s \rightarrow \frac{s}{\Omega_p^{'}}$$

$$H_1(s) = H_p(\frac{\Omega_p}{\Omega_p^{'}} s)$$

Where,

Ω_p= normalized cutoff freq=1 rad/sec

Ω'_p= Desired LP cutoff freq

at $\Omega = \Omega'p$ it is H(j1)

The other transformations are shown in the below table.

Type of Transformation	Transformation	Band edge frequencies of new filter
LP	$S \rightarrow \frac{\Omega_p}{\Omega_p^l} S$	Ω_p'
HP	$S \rightarrow \frac{\Omega_p \Omega_p'}{S}$	Ω_p'
BP	$S \rightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_l - \Omega_u)}$	Ω_l, Ω_u
BS	$S \rightarrow \Omega_p \frac{s(\Omega_u - \Omega_c)}{s^2 + \Omega_u \Omega_l}$	Ω_l, Ω_u

4.3 Frequency Transformation in Digital Domain:

This transformation involves replacing the variable Z^{-1} by a rational function $g(z^{-1})$, while doing this following properties need to be satisfied:

1. ***Mapping Z^{-1} to $g(z^{-1})$ must map points inside the unit circle in the Z- plane onto the unit circle of z- plane to preserve causality of the filter.***

2. ***For stable filter, the inside of the unit circle of the Z - plane must map onto the inside of the unit circle of the z-plane.***

The general form of the function g(.) that satisfy the above requirements of " all-pass " type is

$$g(z^{-1}) = \pm \prod_{k=1}^{n} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}}$$

The different transformations are shown in the below table.

Type of Transformation	Transformation	Parameters
Lowpass	$z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	ω_c=cutoff frequency of new filter $\alpha = \frac{\sin[(\omega'_c - \omega_c)/2]}{\sin[(\omega'_c + \omega_c)/2]}$
Highpass	$z^{-1} \rightarrow -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	ω_c=cutoff frequency of new filter $\alpha = -\frac{\cos[(\omega'_c + \omega_c)/2]}{\cos[(\omega'_c - \omega_c)/2]}$

Bandpass	$z^{-1} \rightarrow -\frac{z^{-2} - \alpha_1 z^{-1} + \alpha_2}{\alpha_2 z^{-2} - \alpha_1 z^{-1} + 1}$	ω_e = lower cutoff frequency ω_u = upper cutoff frequency $\alpha_1 = -2\beta K/(K+1)$ $\alpha_2 = (K-1)/(K+1)$ $\beta = \frac{\cos[(\omega_u + \omega_e)/2]}{\cos[(\omega_u - \omega_e)/2]}$ $K = \cot \frac{\omega_u - \omega_e}{2} \tan \frac{\omega'_c}{2}$
Bandstop	$z^{-1} \rightarrow \frac{z^{-2} - \alpha_1 z^{-1} + \alpha_2}{\alpha_2 z^{-2} - \alpha_1 z^{-1} + 1}$	ω_e = lower cutoff frequency ω_u = upper cutoff frequency $\alpha_1 = -2\beta K/(K+1)$ $\alpha_2 = (K-1)/(K+1)$ $\beta = \frac{\cos[(\omega_u + \omega_e)/2]}{\cos[(\omega_u - \omega_e)/2]}$ $K = \tan \frac{\omega_u - \omega_e}{2} \tan \frac{\omega'_c}{2}$

Prob:

Let $$H(s) = \frac{1}{s^2 + s + 1}$$

Represents the transfer function of a lowpass filter (not butterworth) with a passband of 1 rad/sec. Use freq transformation to find the transfer function of the following filters:

1. ***A LP filter with a passband of 10 rad/sec***
2. ***A HP filter with a cutoff freq of 1 rad/sec***
3. ***A HP filter with a cutoff freq of 10 rad/sec***
4. ***A BP filter with a passband of 10 rad/sec and a corner freq of 100 rad/sec***
5. ***A BS filter with a stopband of 2 rad/sec and a center freq of 10 rad/sec***

Solution:
Given

$$H(s)=\frac{1}{s^2+s+1}$$

a. LP – LP Transform

replace

$$s\rightarrow\frac{s}{\Omega'_p}=\frac{s}{10}$$

$$sub\quad H_a(s)=H(s)|_{s\rightarrow\frac{s}{10}}=\frac{1}{\left((\frac{s}{10})^2+(\frac{s}{10})+1\right)}$$

$$=\frac{100}{s^2+10s+100}$$

b. LP – HP(normalized) Transform

$$s\rightarrow\frac{\Omega_u}{s}=\frac{1}{s}$$

$$sub\quad H_a(s)=H(s)|_{s\rightarrow\frac{1}{s}}=\frac{1}{\left((\frac{1}{s})^2+(\frac{1}{s})+1\right)}$$

$$=\frac{s^2}{s^2+s+1}$$

c. LP – HP(specified cutoff) Transform

replace

$$s \to \frac{\Omega_u}{s} = \frac{10}{s}$$

$$sub \quad H_a(s) = H(s)|_{s \to \frac{10}{s}} = \frac{1}{\left((\frac{10}{s})^2 + (\frac{10}{s}) + 1\right)}$$

$$= \frac{s^2}{s^2 + 10s + 100}$$

d. LP – BP Transform

replace

$$s \to \frac{s^2 + \Omega_u \Omega_l}{s(\Omega_u - \Omega_l)} = \frac{s^2 + \Omega_o^2}{sB_0} \quad where \quad \Omega_o = \sqrt{\Omega_u \Omega_l}$$

$$and \quad B_o = (\Omega_u - \Omega_l)$$

$$sub \quad H_a(s) = H(s)|_{s \to \frac{s^2+10^4}{10s}}$$

$$= \frac{100s^2}{s^4 + 10s^3 + 20100s^2 + 10^5 s + 10^8}$$

e. LP – BS Transform

replace

$$s \to \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l} = \frac{sB_0}{s^2 + \Omega_o^2} \quad where \quad \Omega_o = \sqrt{\Omega_u \Omega_l}$$

$$and \quad B_o = (\Omega_u - \Omega_l)$$

$$sub \quad H_a(s) = H(s)|_{s \to \frac{2s}{s^2+100}}$$

$$= \frac{(s^2+100)^2}{s^4 + 2s^3 + 204s^2 + 200s + 10^4}$$

Prob:

Convert single pole LP Bufferworth filter with system function

$$H(z) = \frac{0.245(1+Z^{-1})}{1+0.509Z^{-1}}$$

into BPF with upper & lower cutoff frequency ω_u & ω_l respectively ,

The LPF has 3-dB bandwidth $\omega_p = 0.2\pi$

Soln: We have the transformation formula given by,

$$z^{-1} \rightarrow \frac{z^{-2} - \alpha_1 z^{-1} + \alpha_2}{a_z z^{-2} - \alpha_1 z^{-1} + 1}$$

applying this to the given transfer function,

$$H(Z) = \frac{0.245\left(1 + \dfrac{z^{-2} - \alpha_1 z^{-1} + \alpha_2}{\alpha_2 z^{-2} - \alpha_1 z^{-1} + 1}\right.}{1 + 0.509\left(\dfrac{z^{-2} - \alpha_1 z^{-1} + \alpha_2}{\alpha_2 z^{-2} - \alpha_1 z^{-1} + 1}\right)}$$

$$H[z] = \frac{0.245(1-\alpha_2)(1-z^{-2})}{(1+0.509\alpha_2) - 1.509\alpha_1 z^{-1} + (\alpha_2 + 0.509)z^{-2}}$$

Note that the resulting filter has zeros at z=±1 and a pair of poles that depend on the choice of ωl and ωu

Ex :

$$\omega_u = \frac{3\pi}{5} \qquad \omega_l = \frac{2\pi}{5}$$

$$\omega_p = 0.2\pi$$

Then k=1, $\alpha_2 = 0, \alpha_1 = 0$

$$\therefore H[z] = \frac{0.245(1 - z^{-2})}{1 + 0.509 z^{-2}}$$

This filter has poles at z=±j0.713 and hence resonates at ω=π/2

The following observations are made,

- It is shown here that how easy to convert one form of filter design to another form.
- What we require is only prototype low pass filter design steps to transform to any other form.

Module -5
Design of FIR Filters

3.1 Introduction:

Two important classes of digital filters based on impulse response type are

Finite Impulse Response (FIR)

Infinite Impulse Response (IIR)

The filter can be expressed in two important forms as:

1) System function representation;

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \qquad (1)$$

2) Difference Equation representation;

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad (2)$$

Each of this form allows various methods of implementation. The eq (2) can be viewed as a computational procedure (an algorithm) for determining the output sequence y(n) of the system from the input sequence x(n). Different realizations are possible with different arrangements of eq (2)

The major issues considered while designing a digital filters are :

- ***Realizability (causal or non causal)***
- ***Stability (filter output will not saturate)***
- ***Sharp Cutoff Characteristics***
- ***Order of the filter need to be minimum (this leads to less delay)***
- ***Generalized procedure (having single procedure for all kinds of filters)***
- ***Linear phase characteristics***

The factors considered with filter implementation are ,

a. It must be a simple design

b. There must be modularity in the implementation so that any order filter can be obtained with lower order modules.

c. Designs must be as general as possible. Having different design procedures for different types of filters(high pass, low pass,…) is cumbersome and complex.

d. Cost of implementation must be as low as possible

e The choice of Software/Hardware realization

3.2 Features of IIR: ***The important features of this class of filters can be listed as:***

- ***Out put is a function of past o/p, present and past i/p's***
- ***It is recursive in nature***
- ***It has at least one Pole (in general poles and zeros)***
- ***Sharp cutoff chas. is achievable with minimum order***
- ***Difficult to have linear phase chas over full range of freq.***
- ***Typical design procedure is analog design then conversion from analog to digital***

3.3 Features of FIR : ***The main features of FIR filter are,***

- ***They are inherently Stable***
- ***Filters with linear phase characteristics can be designed***
- ***Simple implementation – both recursive and nonrecursive structures possible***
- ***Free of limit cycle oscillations when implemented on a finite-word length digital system***

3.3.1 Disadvantages:

- ***Sharp cutoff at the cost of higher order***
- ***Higher order leading to more delay, more memory and higher cost of implementation***

3.4 Importance of Linear Phase:

The group delay is defined as

$$\tau_g = -\frac{d\theta(\omega)}{d\omega}$$

which is negative differential of phase function.

Nonlinear phase results in different frequencies experiencing different delay and arriving at different time at the receiver. This creates problems with speech processing and data communication applications. Having linear phase ensures constant group delay for all frequencies.

The further discussions are focused on FIR filter.

3.5 Examples of simple FIR filtering operations:

1.Unity Gain Filter

$y(n)=x(n)$

2. Constant gain filter

$y(n)=Kx(n)$

3. Unit delay filter

$y(n)=x(n-1)$

4.Two - term Difference filter

$y(n) = x(n)-x(n-1)$

5. Two-term average filter

$y(n) = 0.5(x(n)+x(n-1))$

6. Three-term average filter (3-point moving average filter)

$y(n) = 1/3[x(n)+x(n-1)+x(n-2)]$

7. Central Difference filter

$y(n)= 1/2[x(n) – x(n-2)]$

When we say Order of the filter it is the number of previous inputs used to compute the current output and Filter coefficients are the numbers associated with each of the terms x(n), x(n-1),.. etc

The table below shows order and filter coefficients of above simple filter types:

Ex.	*order*	*a0*	*a1*	*a2*
1	*0*	*1*	-	-
2	*0*	*K*	-	-
3	*1*	*0*	*1*	-
4(HP)	*1*	*1*	*-1*	-
5(LP)	*1*	*1/2*	*1/2*	-
6(LP)	*2*	*1/3*	*1/3*	*1/3*
7(HP)	*2*	*1/2*	*0*	*-1/2*

3.6 Design of FIR filters:

The section to follow will discuss on design of FIR filter. Since linear phase can be achieved with FIR filter we will discuss the conditions required to achieve this.

3.6.1 Symmetric and Antisymmetric FIR filters giving out Linear Phase characteristics:

Symmetry in filter impulse response will ensure Linear phase

An FIR filter of length M with i/p x(n) & o/p y(n) is described by the difference equation:

$$y(n)= b_0\, x(n) + b_1\, x(n-1)+\ldots\ldots+b_{M-1}\, x(n-(M-1)) = \sum_{k=0}^{M-1} b_k x(n-k) \qquad -(1)$$

Alternatively. it can be expressed in convolution form

$$y(n)=\sum_{k=0}^{M-1}h(k)x(n-k) \qquad -(2)$$

i.e b_k= h(k), k=0,1,…..M-1

Filter is also characterized by

$H(z)=\sum_{k=0}^{M-1}h(k)z^{-k}$ -(3) polynomial of degree M-1 in the variable z^{-1}. The roots of this polynomial constitute zeros of the filter.

An FIR filter has linear phase if its unit sample response satisfies the condition
h(n)= ± h(M-1-n) n=0,1,…….M-1 -(4)

Incorporating this symmetry & anti symmetry condition in eq 3 we can show linear phase chas of FIR filters

$$H(z)=h(0)+h(1)z^{-1}+h(2)z^{-2}+\ldots\ldots\ldots\ldots+h(M-2)z^{-(M-2)}+h(M-1)z^{-(M-1)}$$

If M is odd

$$H(z)=h(0)+h(1)z^{-1}+\ldots\ldots\ldots\ldots+h\left(\frac{M-1}{2}\right)z^{-\left(\frac{M-1}{2}\right)}+h\left(\frac{M+1}{2}\right)z^{-\left(\frac{M+1}{2}\right)}+h\left(\frac{M+3}{2}\right)z^{-\left(\frac{M+3}{2}\right)}+\ldots\ldots\ldots\ldots$$
$$+h(M-2)z^{-(M-2)}+h(M-1)z^{-(M-1)}$$

$$=z^{-\left(\frac{M-1}{2}\right)}\left[h(0)z^{\left(\frac{M-1}{2}\right)}+h(1)z^{\left(\frac{M-3}{2}\right)}+\ldots\ldots\ldots\ldots+h\left(\frac{M-1}{2}\right)+h\left(\frac{M+1}{2}\right)z^{-1}+h\left(\frac{M+3}{2}\right)z^{-2}+\ldots\ldots h(M-1)z^{-\left(\frac{M-1}{2}\right)}\right]$$

Applying symmetry conditions for M odd

$$h(0) = \pm h(M-1)$$
$$h(1) = \pm h(M-2)$$
.
.
$$h(\frac{M-1}{2}) = \pm h(\frac{M-1}{2})$$
$$h(\frac{M+1}{2}) = \pm h(\frac{M-3}{2})$$
.
.
$$h(M-1) = \pm h(0)$$

$$H(z) = z^{-(\frac{M-1}{2})}\left[h(\frac{M-1}{2}) + \sum_{n=0}^{\frac{M-3}{2}} h(n)\{z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2}\}\right]$$

similarly for M even

$$H(z) = z^{-(\frac{M-1}{2})}\left[\sum_{n=0}^{\frac{M}{2}-1} h(n)\{z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2}\}\right]$$

3.6.2Frequency response:

If the system impulse response has symmetry property (i.e.,h(n)=h(M-1-n)) and M is odd

$H(e^{j\omega}) = e^{j\theta(\omega)} \,|\, H_r(e^{j\omega}) \,|$ where

$$H_r(e^{j\omega}) = \left[h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega\left(\frac{M-1}{2} - n\right) \right]$$

$$\theta(\omega) = -\left(\frac{M-1}{2}\right)\omega \quad if\ |H_r(e^{j\omega})| \geq 0$$

$$= -\left(\frac{M-1}{2}\right)\omega + \pi \quad if\ |H_r(e^{j\omega})| \leq 0$$

In case of M even the phase response remains the same with magnitude response expressed as

$$H_r(e^{j\omega}) = \left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)\cos\omega\left(\frac{M-1}{2} - n\right) \right]$$

If the impulse response satisfies anti symmetry property (i.e., h(n)=-h(M-1-n))then for M odd we will have

$$h\left(\frac{M-1}{2}\right) = -h\left(\frac{M-1}{2}\right)\ i.e.,\ h\left(\frac{M-1}{2}\right) = 0$$

$$H_r(e^{j\omega}) = \left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)\sin\omega\left(\frac{M-1}{2} - n\right) \right]$$

If M is even then,

$$H_r(e^{j\omega}) = \left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)\sin\omega\left(\frac{M-1}{2} - n\right) \right]$$

In both cases the phase response is given by

$$\theta(\omega) = -(\frac{M-1}{2})\omega + \pi/2 \quad if\ |H_r(e^{j\omega})| \geq 0$$
$$= -(\frac{M-1}{2})\omega + 3\pi/2 \quad if\ |H_r(e^{j\omega})| \leq 0$$

Which clearly shows presence of Linear Phase characteristics.

3.6.3 Comments on filter coefficients:

- ***The number of filter coefficients that specify the frequency response is (M+1)/2 when is M odd and M/2 when M is even in case of symmetric conditions***
- ***In case of impulse response antisymmetric we have h(M-1/2)=0 so that there are (M-1/2) filter coefficients when M is odd and M/2 coefficients when M is even***

3.6.5 Choice of Symmetric and antisymmetric unit sample response

When we have a choice between different symmetric properties, the particular one is picked up based on application for which the filter is used. The following points give an insight to this issue.

- ***If h(n)=-h(M-1-n) and M is odd, $H_r(w)$ implies that Hr(0)=0 & $H_r(\pi)$=0, consequently not suited for lowpass and highpass filter. This condition is suited in Band Pass filter design.***
- ***Similarly if M is even $H_r(0)$=0 hence not used for low pass filter***
- ***Symmetry condition h(n)=h(M-1-n) yields a linear-phase FIR filter with non zero response at w = 0 if desired.***

Looking at these points, antisymmetric properties are not generally preferred.

3.6.6 Zeros of Linear Phase FIR Filters:

Consider the filter system function

$$H(z) = \sum_{n=o}^{M-1} h(n) z^{-n}$$

Expanding this equation

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \ldots\ldots + h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)}$$

$$\text{since for Linear} - \text{phase we need}$$

$$h(n) = h(M-1-n) \quad i.e.,$$

$$h(0) = h(M-1); h(1) = h(M-2); \ldots\ldots h(M-1) = h(0);$$

$$then$$

$$H(z) = h(M-1) + h(M-2)z^{-1} + \ldots\ldots + h(1)z^{-(M-2)} + h(0)z^{-(M-1)}$$

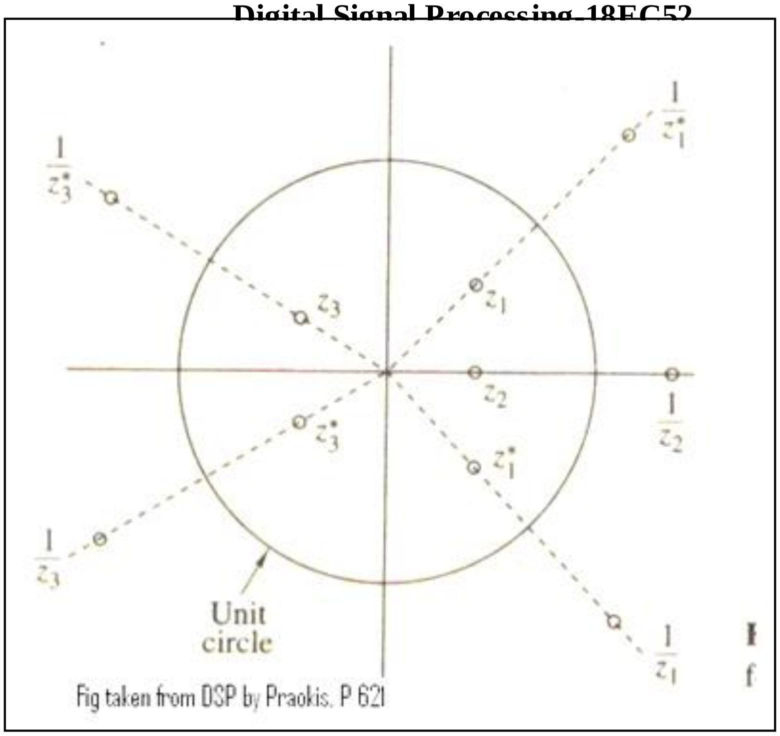
$$H(z) = z^{-(M-1)}[h(M-1)z^{(M-1)} + h(M-2)z^{(M-2)} + \ldots.. + h(1)z + h(0)]$$

$$H(z) = z^{-(M-1)}[\sum_{n=0}^{M-1} h(n)(z^{-1})^{-n}] = z^{-(M-1)} H(z^{-1})$$

This shows that if $z = z_1$ is a zero then $z=z_1^{-1}$ is also a zero

The different possibilities:

1. ***If $z_1 = 1$ then $z_1 = z_1^{-1} = 1$ is also a zero implying it is one zero***
2. ***If the zero is real and $|z|<1$ then we have pair of zeros***
3. ***If zero is complex and $|z|=1$ then and we again have pair of complex zeros.***
4. ***If zero is complex and $|z|\neq 1$ then and we have two pairs of complex zeros***



The plot above shows distribution of zeros for a Linear – phase FIR filter. As it can be seen there is pattern in distribution of these zeros.

3.7 Methods of designing FIR filters:

The standard methods of designing FIR filter can be listed as:

1. ***Fourier series based method***
2. ***Window based method***
3. ***Frequency sampling method***

3.7.1 Design of Linear Phase FIR filter based on Fourier Series method:

Motivation: ***Since the desired freq response $H_d(e^{j\omega})$ is a periodic function in ω with period 2π, it can be expressed as Fourier series expansion***

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}$$

$$\text{where} \quad h_d(n) \quad \text{are} \quad \text{fourier series coefficients}$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

This expansion results in impulse response coefficients which are infinite in duration and non causal. It can be made finite duration by truncating the infinite length. The linear phase can be obtained by introducing symmetric property in the filter impulse response, i.e., h(n) = h(-n). It can be made causal by introducing sufficient delay (depends on filter length)

3.7.2 Stepwise procedure:

1. ***From the desired freq response using inverse FT relation obtain $h_d(n)$***
2. ***Truncate the infinite length of the impulse response to finite length with (assuming M odd)***

$$\begin{aligned} h(n) &= h_d(n) \quad for\ -(M-1)/2 \le n \le (M-1)/2 \\ &= 0 \quad otherwise \end{aligned}$$

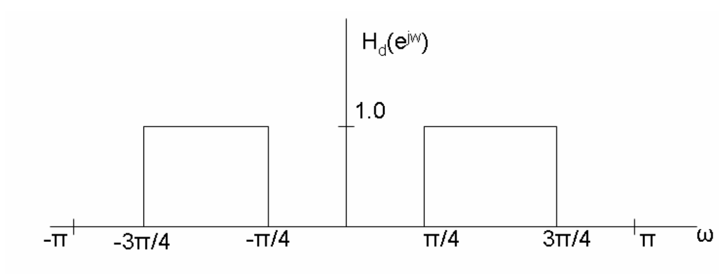
3. ***Introduce h(n) = h(-n) for linear phase characteristics***
4. ***Write the expression for H(z); this is non-causal realization***
5. ***To obtain causal realization $H'(z) = z^{-(M-1)/2} H(z)$***

.

Prob 1 : ***Design an ideal bandpass filter with a frequency response:***

$$H_d(e^{j\omega}) = 1 \qquad for\ \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4}$$
$$= 0 \qquad otherwise$$

Find the values of h(n) for M = 11 and plot the frequency response.



$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$
$$= \frac{1}{2\pi}\left[\int_{-3\pi/4}^{-\pi/4} e^{j\omega n}d\omega + \int_{\pi/4}^{3\pi/4} e^{j\omega n}d\omega\right]$$
$$= \frac{1}{\pi n}\left[\sin\frac{3\pi}{4}n - \sin\frac{\pi}{4}n\right] \quad -\infty \le n \le \infty$$

For n = 0 the value of h(n) is separately evaluated from the basic integration

h(0) = 0.5

Other values of h(n) are evaluated from h(n) expression

h(1)=h(-1)=0

h(2)=h(-2)=-0.3183

h(3)=h(-3)=0

h(4)=h(-4)=0

h(5)=h(-5)=0

The transfer function of the filter is

$$H(z) = h(0) + \sum_{n=1}^{(N-1)/2} [h(n)\{z^n + z^{-n}\}]$$

$$= 0.5 - 0.3183(z^2 + z^{-2})$$

the transfer function of the realizable filter is

$$H^{'}(z) = z^{-5}[0.5 - 0.3183(z^2 + z^{-2})]$$

$$= -0.3183z^{-3} + 0.5z^{-5} - 0.3183z^{-7}$$

the filter coeff are

$$h^{'}(0) = h'(10) = h'(1) = h'(9) = h'(2) = h'(8) = h'(4) = h'(6) = 0$$

$$h'(3) = h'(7) = -0.3183$$

$$h'(5) = 0.5$$

The magnitude response can be expressed as

$$|H(e^{j\omega})| = \sum_{n=1}^{(N-1)/2} a(n)\cos\omega n$$

$$comparing\ this\ \exp\ with$$

$$|H(e^{j\omega})| = |z^{-5}[h(0) + 2\sum_{n=1}^{5} h(n)\cos\omega n]|$$

We have

a(0)=h(0)

a(1)=2h(1)=0

a(2)=2h(2)=-0.6366

a(3)=2h(3)=0

a(4)=2h(4)=0

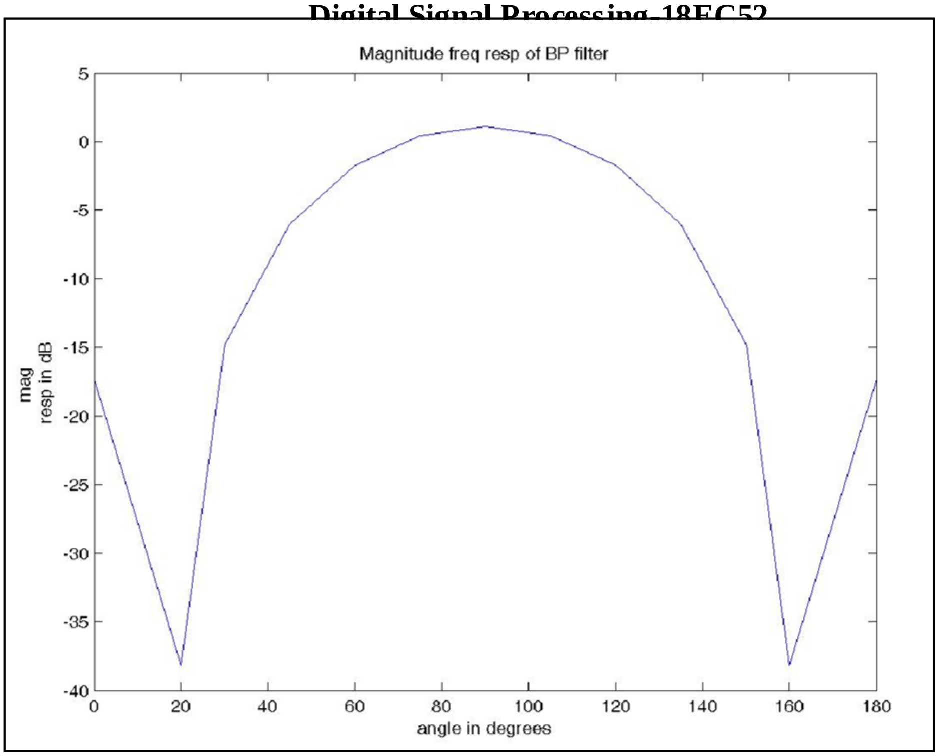
a(5)=2h(5)=0

The magnitude response function is

$|H(e^{j\omega})| = 0.5 - 0.6366 \cos 2\omega$ which can plotted for various values of ω

ω in degrees =[0 20 30 45 60 75 90 105 120 135 150 160 180];

$|H(e^{j\omega})|$ in dBs= [-17.3 -38.17 -14.8 -6.02 -1.74 0.4346 1.11 0.4346 -1.74 -6.02 -14.8 -38.17 -17.3];



Prob 2: ***Design an ideal lowpass filter with a freq response***

$$H_d(e^{j\omega}) = 1 \qquad for -\frac{\pi}{2} \le \omega \le \frac{\pi}{2}$$

$$= 0 \qquad for\ \frac{\pi}{2} \le |\omega| \le \pi$$

Find the values of h(n) for N =11. Find H(z). Plot the magnitude response

From the freq response we can determine $h_d(n)$,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n} d\omega = \frac{\sin\frac{\pi n}{2}}{\pi n} \quad -\infty \le n \le \infty \quad and \quad n \ne 0$$

Truncating $h_d(n)$ to 11 samples

h(0) = 1/2
h(1)=h(-1)=0.3183
h(2)=h(-2)=0
h(3)=h(-3)=-0.106
h(4)=h(-4)=0
h(5)=h(-5)=0.06366

The realizable filter can be obtained by shifting h(n) by 5 samples to right h'(n)=h(n-5)

h'(n)= [0.06366, 0, -0.106, 0, 0.3183, 0.5, 0.3183, 0, -0.106, 0, 0.06366];

$$H'(z) = 0.06366 - 0.106z^{-2} + 0.3183z^{-4} + 0.5z^{-5} + 0.3183z^{-6} - 0.106z^{-8} + 0.06366z^{-10}$$

Using the result of magnitude response for M odd and symmetry

$$H_r(e^{j\omega}) = [h(\frac{M-1}{2}) + \sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega(\frac{M-1}{2} - n)]$$

$$|H_r(e^{j\omega})| = |[0.5 + 0.6366\cos w - 0.212\cos 3w + 0.127\cos 5w]|$$

Exercise Problem:

Design an ideal band reject filter with a frequency response:

$$H_d(e^{j\omega}) = 1 \qquad for|\omega| \le \frac{\pi}{3} \, and \; |\omega| \ge \frac{2\pi}{3}$$
$$= 0 \qquad otherwise$$

Find the values of h(n) for M = 11 and plot the frequency response

Ans:h(n)= [0 -0.1378 0 0.2757 0 0.667 0 0.2757 0 -0.1378 0];

3.8 Window based Linear Phase FIR filter design

The other important method of designing FIR filter is by making use of windows.

The arbitrary truncation of impulse response obtained through inverse Fourier relation can lead to distortions in the final frequency response.The arbitrary truncation is equivalent to multiplying infinite length function with finite length rectangular window, i.e.,

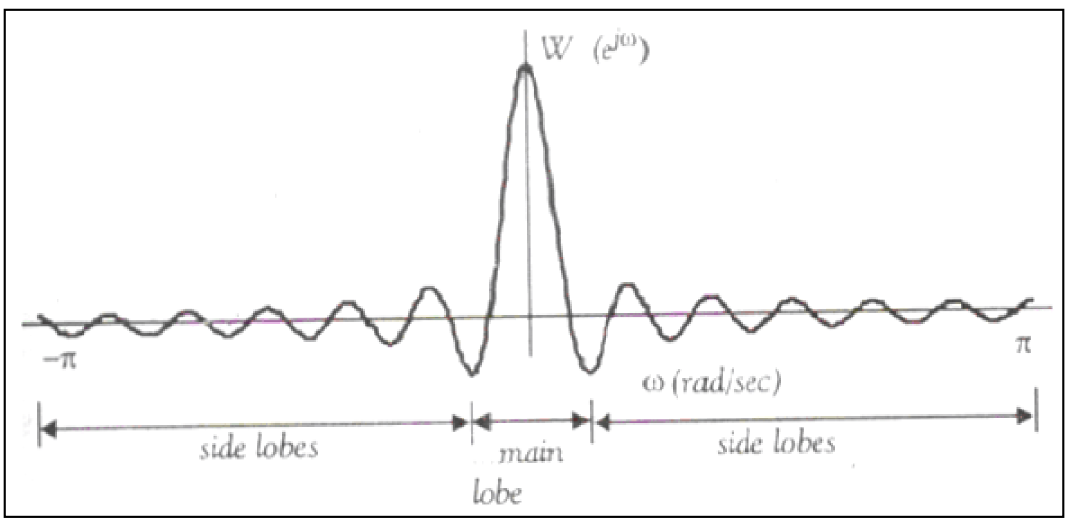
$h(n) = h_d(n)\, w(n)$ where $w(n) = 1$ for $n = \pm(M-1)/2$

The above multiplication in time domain corresponds to convolution in freq domain, i.e.,

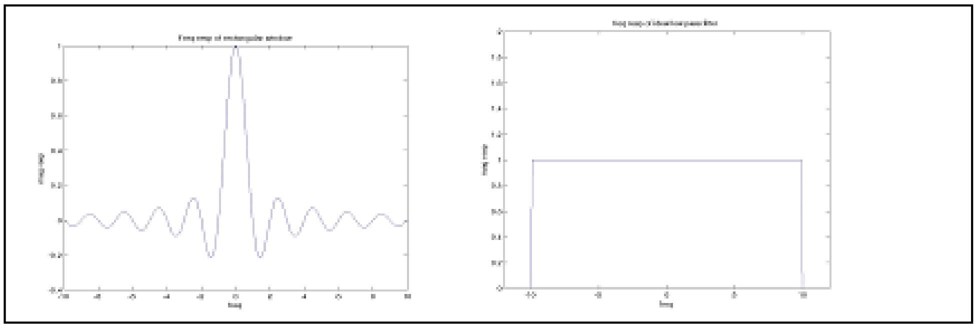
$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$ where $W(e^{j\omega})$ is the FT of window function w(n).

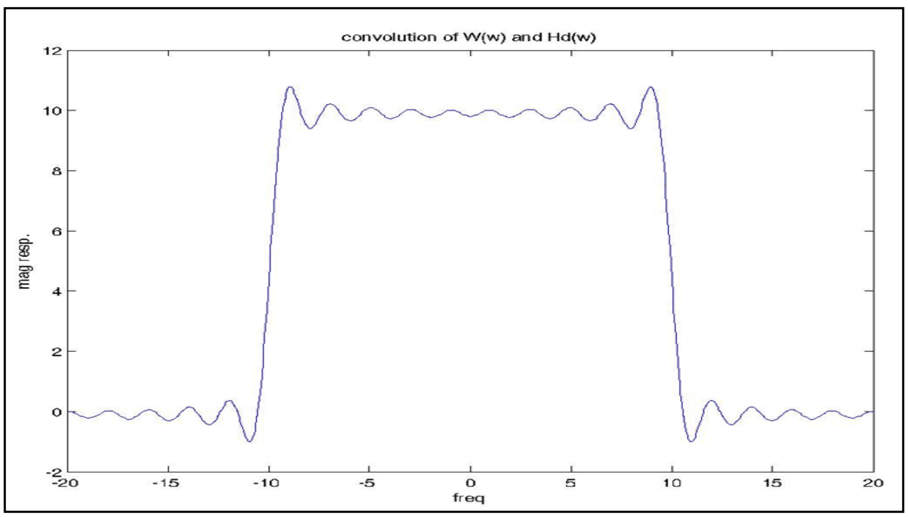
The FT of w(n) is given by

$$W(e^{j\omega}) = \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$



The whole process of multiplying h(n) by a window function and its effect in freq domain are shown in below set of figures.





Suppose the filter to be designed is Low pass filter then the convolution of ideal filter freq response and window function freq response results in distortion in the resultant filter freq response. The ideal sharp cutoff chars are lost and presence of ringing effect is seen at the band edges which is referred to Gibbs Phenomena. This is due to main lobe width and side lobes of the window function freq response.The main lobe width introduces transition band and side lobes results in rippling characters in

pass band and stop band. Smaller the main lobe width smaller will be the transition band. The ripples will be of low amplitude if the peak of the first side lobe is far below the main lobe peak.

3.8.1 How to reduce the distortions?

1. ***Increase length of the window***

- as M increases the main lob width becomes narrower, hence the transition band width is decreased

-With increase in length the side lobe width is decreased but height of each side lobe increases in such a manner that the area under each sidelobe remains invariant to changes in M. Thus ripples and ringing effect in pass-band and stop-band are not changed.

2. ***Choose windows which tapers off slowly rather than ending abruptly***

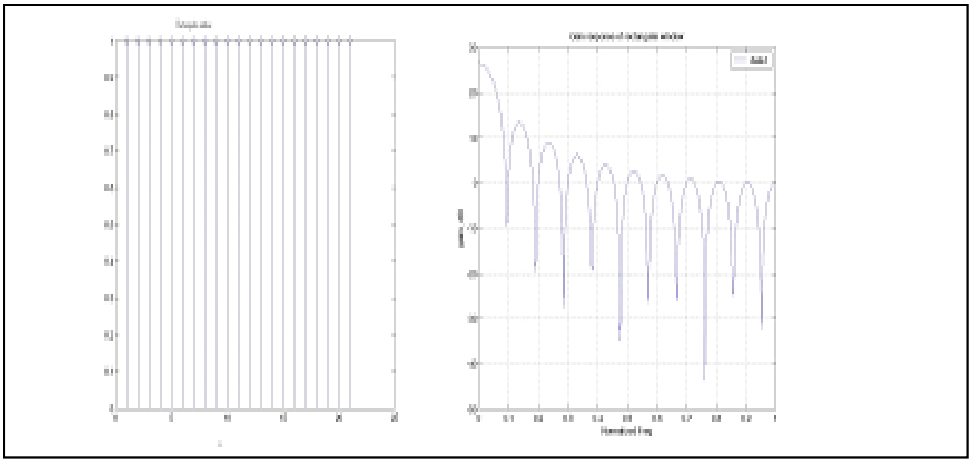
- Slow tapering reduces ringing and ripples but generally increases transition width since main lobe width of these kind of windows are larger.

3.8.2 What is ideal window characteristics?

Window having very small main lobe width with most of the energy contained with it (i.e.,ideal window freq response must be impulsive).Window design is a mathematical problem, more complex the window lesser are the distortions. Rectangular window is one of the simplest window in terms of computational complexity. Windows better than rectangular window are, Hamming, Hanning, Blackman, Bartlett, Traingular,Kaiser. The different window functions are discussed in the following sention.

3.8.3 Rectangular window: ***The mathematical description is given by,***

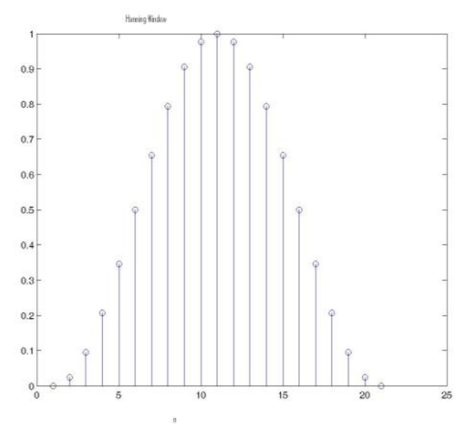
$$w_r(n) = 1 \; for \; 0 \le n \le M-1$$

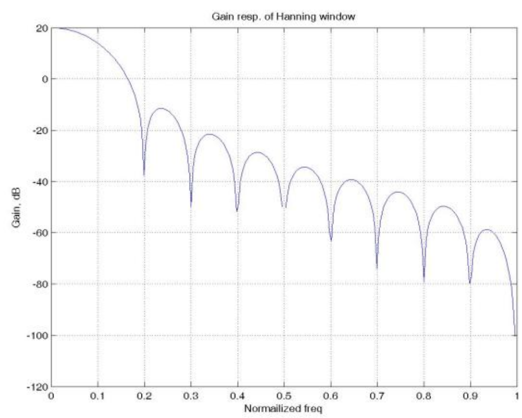


3.8.4 Hanning windows:

It is defined mathematically by,

$$w_{han}(n) = 0.5(1 - \cos\frac{2\pi n}{M-1}) \ \ for \ 0 \le n \le M-1$$

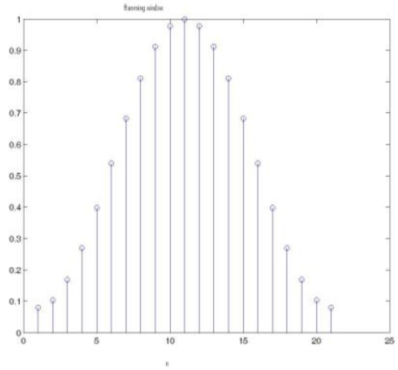


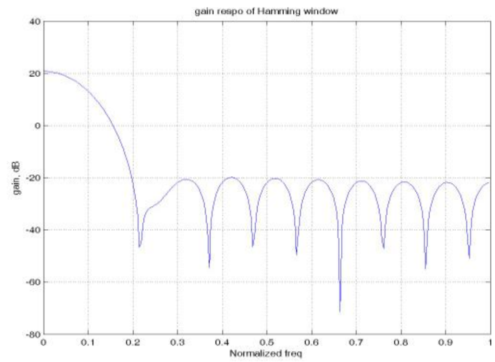


3.8.5 Hamming windows:

This window function is given by,

$$w_{ham}(n) = 0.54 - 0.46\cos\frac{2\pi n}{M-1} \ \ for \ 0 \le n \le M-1$$

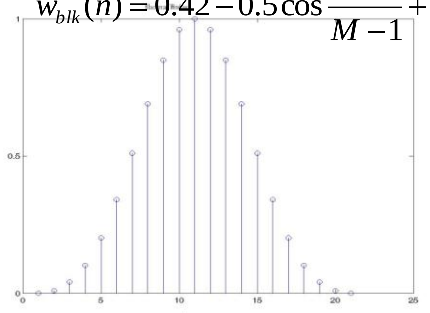


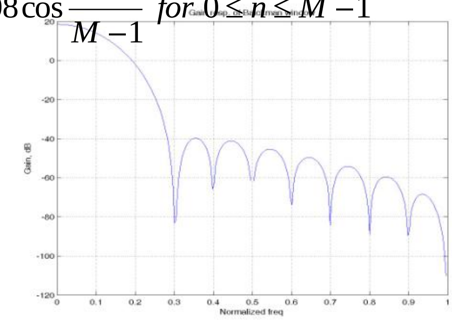


3.8.6 Blackman windows:

This window function is given by,

$$w_{blk}(n) = 0.42 - 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1} \quad for\ 0 \le n \le M-1$$

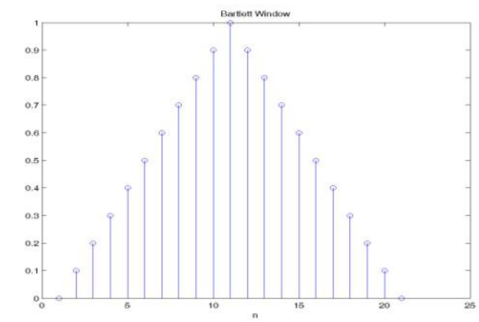


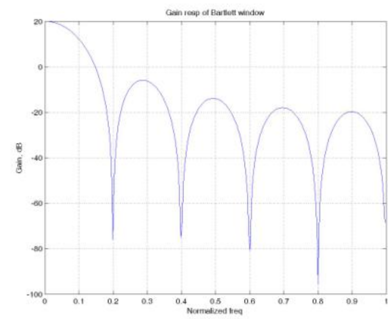


3.8.7 Bartlett (Triangular) windows:

The mathematical description is given by,

$$w_{bart}(n) = 1 - \frac{2\,|\,n - \frac{M-1}{2}\,|}{M-1} \quad for\ 0 \le n \le M-1$$

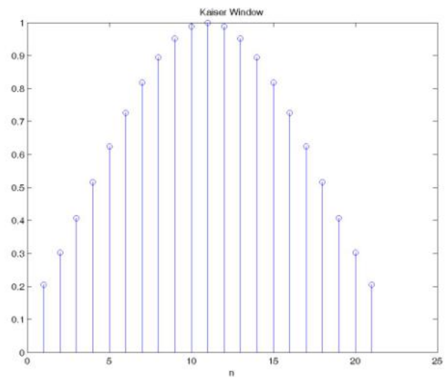


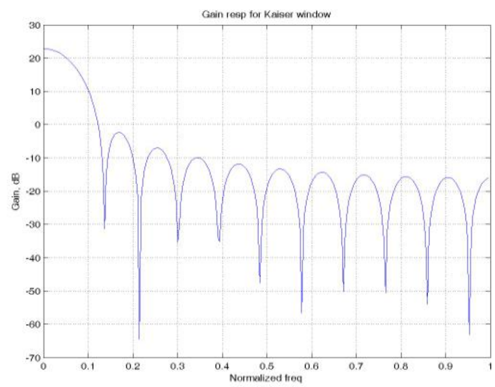


3.8.8 Kaiser windows: ***The mathematical description is given by,***

—

$$w_k(n) = \frac{I_0\left[\alpha\sqrt{\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\frac{M-1}{2}\right)\right]} \qquad \text{for } 0 \le n \le M-1$$





Type of window	*Appr. Transition width of the main lobe*	*Peak sidelobe (dB)*
Rectangular	***$4\pi/M$***	***-13***
Bartlett	***$8\pi/M$***	***-27***
Hanning	***$8\pi/M$***	***-32***
Hamming	***$8\pi/M$***	***-43***
Blackman	***$12\pi/M$***	***-58***

Looking at the above table we observe filters which are mathematically simple do not offer best characteristics. Among the window functions discussed Kaiser is the most complex one in terms of functional description whereas it is the one which offers maximum flexibility in the design.

3.8.9 Procedure for designing linear-phase FIR filters using windows:

1. ***Obtain $h_d(n)$ from the desired freq response using inverse FT relation***
2. ***Truncate the infinite length of the impulse response to finite length with***

(assuming M odd) choosing proper window

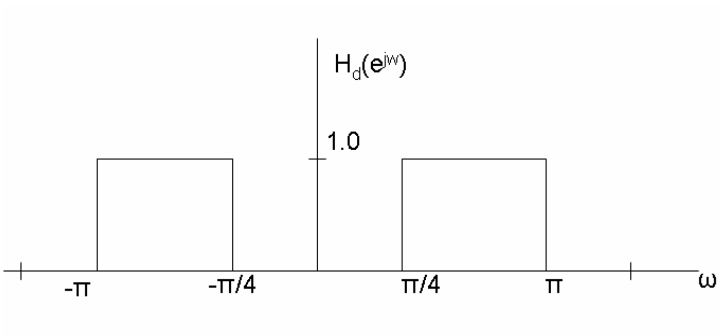
$$h(n) = h_d(n)w(n) \;\; where$$
$$w(n) \; is \; the \; window \; function \; defined \;\; for \; -(M-1)/2 \le n \le (M-1)/2$$

3. ***Introduce h(n) = h(-n) for linear phase characteristics***
4. ***Write the expression for H(z); this is non-causal realization***
5. ***To obtain causal realization H'(z) = $z^{-(M-1)/2}$ H(z)***

Prob 1: ***Design an ideal highpass filter with a frequency response:***

$$H_d(e^{j\omega}) = 1 \qquad for \; \frac{\pi}{4} \le |\omega| \le \pi$$
$$= 0 \qquad |\omega| < \frac{\pi}{4}$$

using a hanning window with M = 11 and plot the frequency response.



$$h_d(n) = \frac{1}{2\pi}[\int_{-\pi}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{\pi} e^{j\omega n} d\omega]$$

$$h_d(n) = \frac{1}{\pi n}[\sin \pi n - \sin \frac{\pi n}{4}] \quad for \quad -\infty \le n \le \infty \quad and \quad n \ne 0$$

$$h_d(0) = \frac{1}{2\pi}[\int_{-\pi}^{-\pi/4} d\omega + \int_{\pi/4}^{\pi} d\omega] = \frac{3}{4} = 0.75$$

$h_d(1) = h_d(-1)$=-0.225
$h_d(2) = h_d(-2)$= -0.159
$h_d(3) = h_d(-3)$= -0.075
$h_d(4) = h_d(-4)$= 0
$h_d(5) = h_d(-5)$ = 0.045

The hamming window function is given by

$$w_{hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{M-1} \qquad -(\frac{M-1}{2}) \le n \le (\frac{M-1}{2})$$
$$= 0 \qquad otherwise$$

$$for \quad N = 11$$
$$w_{hn}(n) = 0.5 + 0.5\cos\frac{\pi n}{5} \qquad -5 \le n \le 5$$

$w_{hn}(0) = 1$
$w_{hn}(1) = w_{hn}(-1)=0.9045$
$w_{hn}(2)= w_{hn}(-2)=0.655$
$w_{hn}(3)= w_{hn}(-3)= 0.345$
$w_{hn}(4)= w_{hn}(-4)=0.0945$
$w_{hn}(5)= w_{hn}(-5)=0$

h(n)= $w_{hn}(n)h_d(n)$

h(n)=[0 0 -0.026 -0.104 -0.204 0.75 -0.204 -0.104 -0.026 0 0]

$$h'(n) = h(n-5)$$
$$H'(z) = -0.026z^{-2} - 0.104z^{-3} - 0.204z^{-4} + 0.75z^{-5} - 0.204z^{-6} - 0.104z^{-7} - 0.026z^{-8}$$

Using the equation

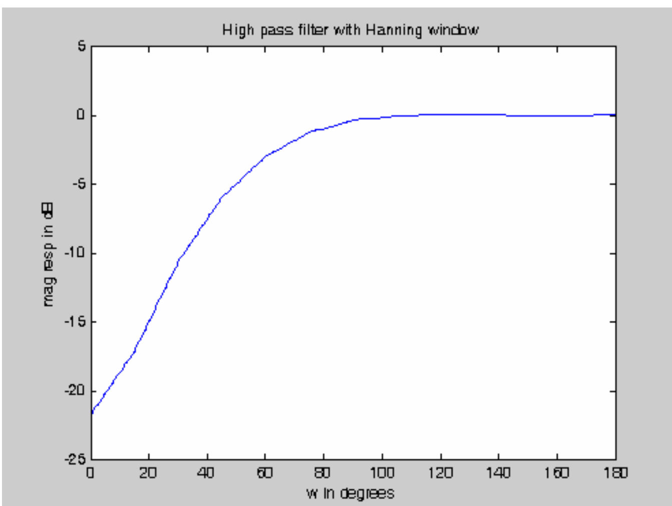
$$H_r(e^{jw}) = [h(\frac{M-1}{2}) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega(\frac{M-1}{2} - n)$$
$$H_r(e^{jw}) = 0.75) + 2\sum_{n=0}^{4} h(n)\cos\omega(5-n)$$

The magnitude response is given by,

$$|Hr(e^{j\omega})| = |0.75 - 0.408\cos\omega - 0.208\cos 2\omega - 0.052\cos 3\omega|$$

ω in degrees = [0 15 30 45 60 75 90 105 120 135 150 165 180]

$|H(e^{j\omega})|$ in dBs = [-21.72 -17.14 -10.67 -6.05 -3.07 -1.297 -0.3726 -0.0087 0.052 0.015 0 0 0.017]



Prob 2 : **Design a filter with a frequency response:**

$$H_d(e^{j\omega}) = e^{-j3\omega} \qquad for -\frac{\pi}{4} \le \omega \le \frac{\pi}{4}$$

$$= 0 \quad \frac{\pi}{4} < |\omega| \le \pi$$

using a Hanning window with M = 7

Soln:

The freq resp is having a term e $^{-j\omega(M-1)/2}$ which gives h(n) symmetrical about n = M-1/2 = 3 i.e we get a causal sequence.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j3\omega} e^{j\omega n} d\omega$$

$$= \frac{\sin\frac{\pi}{4}(n-3)}{\pi(n-3)}$$

$$this\ gives\ h_d(0) = h_d(6) = 0.075$$

$$h_d(1) = h_d(5) = 0.159$$

$$h_d(2) = h_d(4) = 0.22$$

$$h_d(3) = 0.25$$

The Hanning window function values are given by

$w_{hn}(0) = w_{hn}(6) = 0$

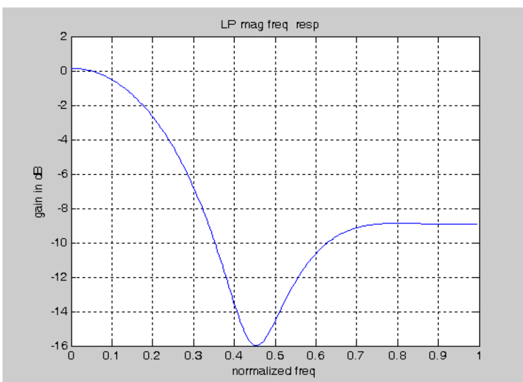
$w_{hn}(1)= w_{hn}(5) =0.25$

$w_{hn}(2)= w_{hn}(4) =0.75$

$w_{hn}(3)=1$

$h(n)=h_d(n)\ w_{hn}(n)$

$h(n)=[0\ \ 0.03975\ \ 0.165\ \ 0.25\ \ 0.165\ \ 0.3975\ \ 0]$



3.9 Design of Linear Phase FIR filters using Frequency Sampling method:

3.9.1 Motivation: ***We know that DFT of a finite duration DT sequence is obtained by sampling FT of the sequence then DFT samples can be used in reconstructing original time domain samples if frequency domain sampling was done correctly. The samples of FT of h(n) i.e., H(k) are sufficient to recover h(n).***

Since the designed filter has to be realizable then h(n) has to be real, hence even symmetry properties for mag response |H(k)| and odd symmetry properties for phase response can be applied. Also, symmetry for h(n) is applied to obtain linear phase chas.

Fro DFT relationship we have

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j2\pi kn/N} \quad for \quad n = 0,1,......N-1$$

$$H(k) = \sum_{n=0}^{N-1} h(n)e^{-j2\pi kn/N} \quad for \quad k = 0,1,.........N-1$$

Also we know $H(k) = H(z)|_{z=e^{j2\pi kn/N}}$

The system function H(z) is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}$$

Substituting for h(n) from IDFT relationship

$$H(z) = \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{H(k)}{1-e^{j2\pi kn/N}z^{-1}}$$

Since H(k) is obtained by sampling $H(e^{j\omega})$ hence the method is called **Frequency Sampling Technique.**

Since the impulse response samples or coefficients of the filter has to be real for filter to be realizable with simple arithmetic operations, properties of DFT of real sequence can be used. The following properties of DFT for real sequences are useful:

$H^*(k) = H(N-k)$

$|H(k)|=|H(N-k)|$ - magnitude response is even

$\theta(k) = - \theta(N-k)$ – Phase response is odd

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j2\pi kn/N}$$ can be rewritten as (for N odd)

$$h(n) = \frac{1}{N}\left[H(0) + \sum_{k=1}^{N-1} H(k)e^{j2\pi kn/N}\right]$$

$$h(n) = \frac{1}{N}\left[H(0) + \sum_{k=1}^{N-1/2} H(k)e^{j2\pi kn/N} + \sum_{k=N-1/2}^{N-1} H(k)e^{j2\pi kn/N}\right]$$

Using substitution k = N – r or r = N- k in the second substitution
with r going from now (N- 1)/2 to 1 as k goes from 1 to (N-1)/2

$$h(n) = \frac{1}{N}\left[H(0) + \sum_{k=1}^{(N-1)/2} H(k)e^{j2\pi kn/N} + \sum_{k=1}^{(N-1)/2} H(N-k)e^{-j2\pi kn/N}\right]$$

$$h(n) = \frac{1}{N}\left[H(0) + \sum_{k=1}^{(N-1)/2} H(k)e^{j2\pi kn/N} + \sum_{k=1}^{(N-1)/2} H^*(k)e^{-j2\pi kn/N}\right]$$

$$h(n) = \frac{1}{N}\left[H(0) + \sum_{k=1}^{(N-1)/2} H(k)e^{j2\pi kn/N} + \sum_{k=1}^{(N-1)/2} (H(k)e^{j2\pi kn/N})^*\right]$$

$$h(n) = \frac{1}{N}\left[H(0) + \sum_{k=1}^{(N-1)/2} (H(k)e^{j2\pi kn/N} + (H(k)e^{j2\pi kn/N})^*\right]$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{(N-1)/2} \mathrm{Re}(H(k)e^{j2\pi kn/N}\right]$$

Similarly for N even we have

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{(N-1)/2} \mathrm{Re}(H(k)e^{j2\pi kn/N}\right]$$

Using the symmetry property h(n)= h (N-1-n) we can obtain Linear phase FIR filters using the frequency sampling technique.

Prob 1 : ***Design a LP FIR filter using Freq sampling technique having cutoff freq of π/2 rad/sample. The filter should have linear phase and length of 17.***

The desired response can be expressed as

$$H_d(e^{j\omega}) = e^{-j\omega(\frac{M-1}{2})} \quad for \quad |\omega| \le \omega c$$
$$= 0 \quad otherwise$$
$$with \quad M = 17 \quad and \quad \omega c = \pi/2$$

$$H_d(e^{j\omega}) = e^{-j\omega 8} \quad for \quad 0 \le \omega \le \pi/2$$
$$= 0 \quad for \quad \pi/2 \le \omega \le \pi$$

Selecting $\omega_k = \dfrac{2\pi k}{M} = \dfrac{2\pi k}{17} \quad for \quad k = 0,1,......16$

$$H(k) = H_d(e^{j\omega})|_{\omega=\frac{2\pi k}{17}}$$

$$H(k) = e^{-j\frac{2\pi k}{17}8} \quad for \quad 0 \le \frac{2\pi k}{17} \le \frac{\pi}{2}$$
$$= 0 \quad for \quad \pi/2 \le \frac{2\pi k}{17} \le \pi$$
$$H(k) = e^{-j\frac{16\pi k}{17}} \quad for \quad 0 \le k \le \frac{17}{4}$$
$$= 0 \quad for \quad \frac{17}{4} \le k \le \frac{17}{2}$$

The range for "k" can be adjusted to be an integer such as

$$0 \le k \le 4$$
$$and \quad 5 \le k \le 8$$

The freq response is given by

$$H(k) = e^{-j\frac{2\pi k}{17}8} \qquad for \quad 0 \le k \le 4$$
$$= 0 \qquad for \qquad 5 \le k \le 8$$

Using these value of H(k) we obtain h(n) from the equation

$$h(n) = \frac{1}{M}(H(0) + 2\sum_{k=1}^{(M-1)/2} \mathrm{Re}(H(k)e^{j2\pi kn/M}))$$

i.e.,

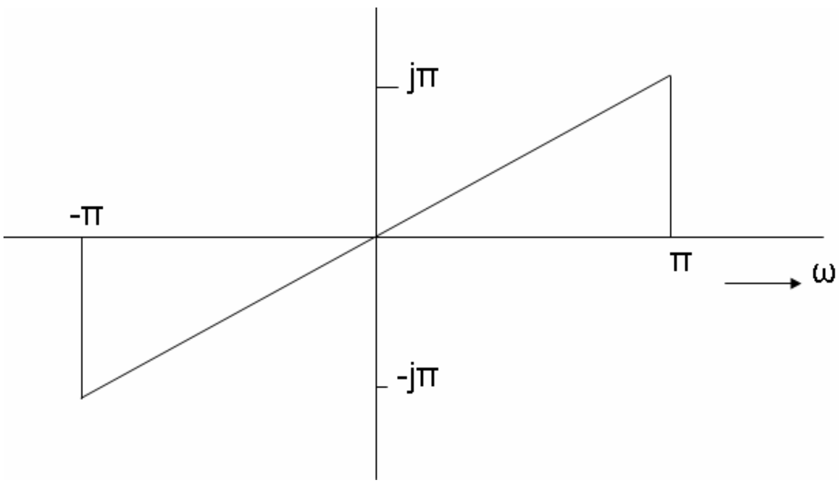
$$h(n) = \frac{1}{17}(1 + 2\sum_{k=1}^{4} \mathrm{Re}(e^{-j16\pi k/17}e^{j2\pi kn/17}))$$

$$h(n) = \frac{1}{17}(H(0) + 2\sum_{k=1}^{4} \cos(\frac{2\pi k(8-n)}{17}) \quad for \quad n = 0,1,........16$$

- Even though k varies from 0 to 16 since we considered ω varying between 0 and π/2 only k values from 0 to 8 are considered
- While finding h(n) we observe symmetry in h(n) such that n varying 0 to 7 and 9 to 16 have same set of h(n)

3.10 Design of FIR Differentiator

Differentiators are widely used in Digital and Analog systems whenever a derivative of the signal is needed. Ideal differentiator has pure linear magnitude response in the freq range –π to +π. The typical frequency response characteristics is as shown in the below figure.



Prob: Design an Ideal Differentiator using a) rectangular window and b)Hamming window with length of the system = 7.

Solution:
As seen from differentiator frequency chars. It is defined as

$H(e^{j\omega}) = j\omega$ between $-\pi$ to $+\pi$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega\, e^{j\omega n}\, d\omega = \frac{\cos \pi n}{n} \qquad -\infty \le n \le \infty \quad and \quad n \ne 0$$

The $h_d(n)$ is an add function with $h_d(n)=-h_d(-n)$ and $h_d(0)=0$

a) rectangular window

$h(n)=h_d(n)w_r(n)$

h(1)=-h(-1)=hd(1)=-1
h(2)=-h(-2)=hd(2)=0.5
h(3)=-h(-3)=hd(3)=-0.33

h'(n)=h(n-3) for causal system
thus,

$$H'(z) = 0.33 - 0.5z^{-1} + z^{-2} - z^{-4} + 0.5z^{-5} - 0.33z^{-6}$$

Also from the equation

$$H_r(e^{j\omega}) = 2\sum_{n=0}^{(M-3)/2} h(n)\sin\omega(\frac{M-1}{2} - n)$$

For M=7 and h'(n) as found above we obtain this as

$$H_r(e^{j\omega}) = 0.66\sin 3\omega - \sin 2\omega + 2\sin\omega$$

$$H(e^{j\omega}) = jH_r(e^{j\omega}) = j(0.66\sin 3\omega - \sin 2\omega + 2\sin\omega)$$

b) Hamming window
$h(n)=h_d(n)w_h(n)$

where $w_h(n)$ is given by

$$w_h(n) = 0.54 + 0.46\cos\frac{2\pi n}{(M-1)} \quad -(M-1)/2 \le n \le (M-1)/2$$
$$= 0 \quad otherwise$$

For the present problem

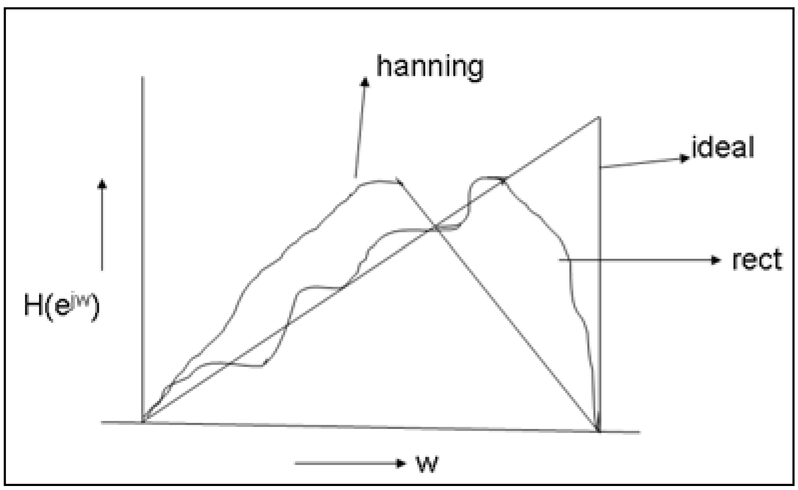
$$w_h(n) = 0.54 + 0.46\cos\frac{\pi n}{3} \quad -3 \le n \le 3$$

The window function coefficients are given by for n=-3 to +3
Wh(n)= [0.08 0.31 0.77 1 0.77 0.31 0.08]

Thus h'(n) = h(n-5) = [0.0267, -0.155, 0.77, 0, -0.77, 0.155, -0.0267]

Similar to the earlier case of rectangular window we can write the freq response of differentiator as

$$H(e^{j\omega}) = jH_r(e^{j\omega}) = j(0.0534\sin 3\omega - 0.31\sin 2\omega + 1.54\sin\omega)$$



We observe

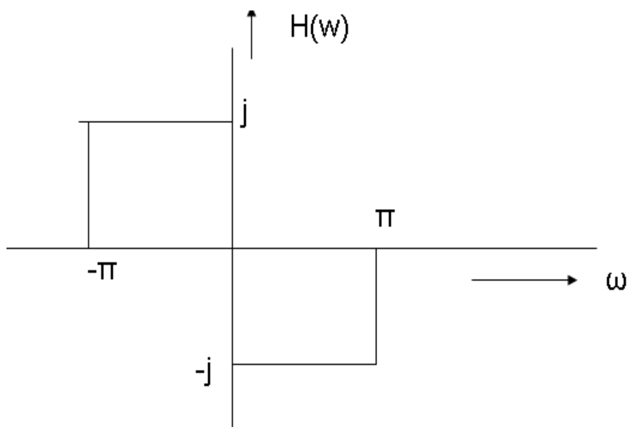
- With rectangular window, the effect of ripple is more and transition band width is small compared with hamming window
- With hamming window, effect of ripple is less whereas transition band is more

3.11 Design of FIR Hilbert transformer:

Hilbert transformers are used to obtain phase shift of 90 degree. They are also called j operators. They are typically required in quadrature signal processing. The Hilbert

transformer is very useful when out of phase component (or imaginary part) need to be generated from available real component of the signal.

Prob: Design an ideal Hilbert transformer using a) rectangular window and b) Blackman Window with M = 11



Solution:

As seen from freq chars it is defined as

$$H_d(e^{j\omega}) = j \quad -\pi \le \omega \le 0$$
$$= -j \qquad 0 \le \omega \le \pi$$

The impulse response is given by

$$h_d(n) = \frac{1}{2\pi}[\int_{-\pi}^{0} je^{j\omega n}d\omega + \int_{0}^{\pi} -je^{j\omega n}d\omega] = \frac{(1-\cos\pi n)}{\pi n} \quad -\infty \le n \le \infty \quad except \quad n = 0$$

At n = 0 it is hd(0) = 0 and hd(n) is an odd function

a) Rectangular window
$h(n) = h_d(n)\, w_r(n) = h_d(n)$ for $-5 \geq n \geq 5$

$h'(n)=h(n-5)$

$h(n)= [-0.127, 0, -0.212, 0, -0.636, 0, 0.636, 0, 0.212, 0, 0.127]$

$$H_r(e^{j\omega}) = 2\sum_{n=0}^{4} h(n)\sin\omega(5-n)$$

$$H(e^{j\omega}) = j\,|H_r(e^{j\omega})| = j\{0.254\sin 5\omega + 0.424\sin 3\omega + 1.272\sin\omega\}$$

b) Blackman Window
window function is defined as

$$w_b(n) = 0.42 + 0.5\cos\frac{\pi n}{5} + 0.08\cos\frac{2\pi n}{5} \qquad -5 \leq n \leq 5$$
$$= 0 \qquad otherwise$$

$W_b(n) = [0, 0.04, 0.2, 0.509, 0.849, 1, 0.849, 0.509, 0.2, 0.04, 0]$ for $-5 \geq n \geq 5$

$h'(n) = h(n-5) = [0, 0, -0.0424, 0, -0.5405, 0, 0.5405, 0, 0.0424, 0, 0]$

$$H(e^{j\omega}) = -j[0.0848\sin 3\omega + 1.0810\sin\omega]$$